0 Vector and Tensor Algebra

0.1 Vectors and the Summation Convention

In order to describe physical phenomena in space, it is necessary to define a frame of reference. For our purposes, we first introduce an orthonormal, time-invariant basis (also known as the Cartesian basis) in the three-dimensional Euclidean vector space $\mathbb{R}^3$ be denoted by

$$\mathcal{B} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3\}\,, \tag{1}$$

such that each vector $\boldsymbol{v}$ with components v_i $(i = 1 \ldots 3)$ can be expressed in its basis representation

$$\boldsymbol{v} = \sum_{i=1}^{3} v_i\, \boldsymbol{g}_i \tag{2}$$

or, using Einstein's summation convention, as

$$\boldsymbol{u} = v_i\, \boldsymbol{g}_i. \tag{3}$$

The *summation convention* implies that repeated indices appearing exactly twice in a single term (by term we mean any single vector or tensor quantity or any product of vector and/or tensor quantities) are to be summed from 1 through d (where d is the number of dimensions), i.e., for example in three dimensions,

$$\boldsymbol{v} = v_i\, \boldsymbol{g}_i = v_1\, \boldsymbol{g}_1 + v_2\, \boldsymbol{g}_2 + v_3\, \boldsymbol{g}_3. \tag{4}$$

As a common convention, one uses Greek letters for summations in two dimensions (e.g., in plane strain in problems of solid mechanics). Therefore, in two dimensions one writes

$$\boldsymbol{v} = v_\alpha\, \boldsymbol{g}_\alpha = v_1\, \boldsymbol{g}_1 + v_2\, \boldsymbol{g}_2. \tag{5}$$

A repeated index to be summed is called a *dummy index*, while those indices only appearing once (and hence not requiring summation) are known as *free indices*. To avoid ambiguity, no index is allowed to appear more than twice in in a single term. For example, the definition

$$\boldsymbol{a} = \sum_{i=1}^{3} \lambda_i\, \gamma_i\, \boldsymbol{g}_i \tag{6}$$

must not be abbreviated as $\boldsymbol{a} = \lambda_i\, \gamma_i\, \boldsymbol{g}_i$.

When using index notation, it is tedious to write out the base vectors for every vector quantity. Instead one often writes v_i for short, which implies (due to the single free index) that we are dealing with a vector quantity whose components are v_i and which reads in full $v_i\, \boldsymbol{g}_i$. A vector equation such as $\boldsymbol{u} = \boldsymbol{v}$ then becomes $u_i = v_i$ and in fact denotes three equations $(i = 1, 2, 3)$. Here and in the following, we usually denote vector quantities by lower case bold letters.

Finally, note that vectors $\boldsymbol{v}$ exist independently of the choice of the coordinate system, i.e., in symbolic notation we write $\boldsymbol{v}$ to identify a specific vector. Its components v_i, however, depend on the choice of the reference frame and hence are not invariant under, e.g., coordinate transformations. In conclusion, symbolic notation is frame-independent, use of components (such as when using indicial notation) requires knowledge of the specific coordinate system in which they are defined.

0.2 Inner Product of Two Vectors

We define the inner product of two vectors in the following way:

$$\boldsymbol{u} \cdot \boldsymbol{v} = u\, v\, \cos\varphi(\boldsymbol{u}, \boldsymbol{v}) \tag{7}$$

with $\varphi(\boldsymbol{u}, \boldsymbol{v})$ the angle subtended by $\boldsymbol{u}$ and $\boldsymbol{v}$, and u and v the lengths of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively. Applying this definition to the unit vectors $\boldsymbol{g}_i$ shows that

$$\boldsymbol{g}_i \cdot \boldsymbol{g}_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{8}$$

if $\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$. For convenience, we introduce the *Kronecker delta* for the orthonormal basis as

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \tag{9}$$

which leads to

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i\, \boldsymbol{g}_i \cdot v_j\, \boldsymbol{g}_j = u_i\, v_j\, (\boldsymbol{g}_i \cdot \boldsymbol{g}_j) = u_i\, v_j\, \delta_{ij} = u_i\, v_i. \tag{10}$$

Therefore, the resulting quantity of the inner product of two vectors is a scalar (and the inner product of two vectors is often referred to as *scalar product*). Geometrically, $\boldsymbol{u} \cdot \boldsymbol{v}$ represents the length of vector $\boldsymbol{u}$ projected onto $\boldsymbol{v}$ or vice-versa. In particular, we conclude that the inner product of vector $\boldsymbol{v}$ with one of the basis vectors $\boldsymbol{g}_i$ in an orthonormal frame yields the projection of $\boldsymbol{v}$ onto $\boldsymbol{g}_i$, which is the component of $\boldsymbol{v}$ in the direction of coordinate i:

$$\boldsymbol{v} \cdot \boldsymbol{g}_i = v_j\, \boldsymbol{g}_j \cdot \boldsymbol{g}_i = v_j\, \delta_{ij} = v_i. \tag{11}$$

As a further consequence, the inner product of orthogonal vectors vanishes:

$$\boldsymbol{u} \perp \boldsymbol{v} \quad \Leftrightarrow \quad \boldsymbol{u} \cdot \boldsymbol{v} = 0. \tag{12}$$

By the aid of the above definition, we can introduce the *norm* of a vector in a Cartesian reference frame as

$$u = |\boldsymbol{u}| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}} = \sqrt{u_i\, u_i}. \tag{13}$$

It will be helpful in subsequent sections to realize that

$$\delta_{ii} = 3. \tag{14}$$

0.3 Cross Product of Two Vectors

Let us define the cross product of two basis vectors in an orthonormal basis by

$$\boldsymbol{u} \times \boldsymbol{v} = u\, v\, \sin\varphi(\boldsymbol{u}, \boldsymbol{v})\, \boldsymbol{g}_\perp, \tag{15}$$

where $\varphi(\boldsymbol{u}, \boldsymbol{v})$ is again the angle subtended by $\boldsymbol{u}$ and $\boldsymbol{v}$, and $\boldsymbol{g}_\perp$ is a unit vector perpendicular to the plane spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$ and oriented by the right-hand-rule. Geometrically,

$$|\boldsymbol{u} \times \boldsymbol{v}| = u\, v\, \sin\varphi(\boldsymbol{u}, \boldsymbol{v}) \tag{16}$$

represents the area of the parallelpiped spanned by vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. By applying the above definition to the unit vectors in an orthonormal basis, we see that

$$\boldsymbol{g}_1 \times \boldsymbol{g}_2 = \boldsymbol{g}_3, \qquad \boldsymbol{g}_2 \times \boldsymbol{g}_3 = \boldsymbol{g}_1, \qquad \boldsymbol{g}_3 \times \boldsymbol{g}_1 = \boldsymbol{g}_2 \tag{17}$$

$$\boldsymbol{g}_2 \times \boldsymbol{g}_1 = -\boldsymbol{g}_3, \qquad \boldsymbol{g}_3 \times \boldsymbol{g}_2 = -\boldsymbol{g}_1, \qquad \boldsymbol{g}_1 \times \boldsymbol{g}_3 = -\boldsymbol{g}_2 \tag{18}$$

$$\boldsymbol{g}_1 \times \boldsymbol{g}_1 = 0, \qquad \boldsymbol{g}_2 \times \boldsymbol{g}_2 = 0, \qquad \boldsymbol{g}_3 \times \boldsymbol{g}_3 = 0. \tag{19}$$

Note that the last line of these equations cannot be abbreviated in the standard manner using summation convention: writing $\boldsymbol{g}_i \times \boldsymbol{g}_i = 0$ would imply summation over i (although we only want to express that the relations holds for all $i = 1, \ldots, 3$). When summation must be suppressed like in this case, one writes

$$\boldsymbol{g}_{(i)} \times \boldsymbol{g}_{(i)} = 0, \tag{20}$$

which signifies that the above relation holds for every i and no summation over i is implied.

The relations given above for the cross product of two basis vectors can be abbreviated by

$$\boldsymbol{g}_i \times \boldsymbol{g}_j = \epsilon_{ijk}\, \boldsymbol{g}_k \tag{21}$$

where we introduced the *permutation symbol* (sometimes called the Levi-Civita symbol)

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } i,j,k \text{ is a cyclic (even) sequence,} \\ -1, & \text{if } i,j,k \text{ is an anticyclic (odd) sequence,} \\ 0, & \text{if } i,j,k \text{ is an acyclic sequence (at least one index appears more than once).} \end{cases} \tag{22}$$

For example, $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, $\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$ and all other combinations vanish. As a consequence, the cross product of two vectors becomes

$$\boldsymbol{u} \times \boldsymbol{v} = u_i\, v_j\, \boldsymbol{g}_i \times \boldsymbol{g}_j = \epsilon_{ijk}\, u_i\, v_j\, \boldsymbol{g}_k. \tag{23}$$

Following the aforementioned convention of omitting the basis vectors in an orthonormal basis, we can express this in component form as

$$(\boldsymbol{u} \times \boldsymbol{v})_k = \epsilon_{ijk}\, u_i\, v_j \tag{24}$$

and summation convention implies that the result is a vector quantity (k is only free index, while i, j are dummy indices subject to summation). Therefore, the cross product of two vectors is often referred to as *vector product*.

Note that the order of indices (i, j, k) can be varied arbitrarily as long as it remains cyclic / anticyclic, e.g.

$$\epsilon_{ijk}\, u_i\, v_j\, \boldsymbol{g}_k = \epsilon_{jki}\, u_i\, v_j\, \boldsymbol{g}_k = \epsilon_{kij}\, u_i\, v_j\, \boldsymbol{g}_k = \epsilon_{ijk}\, u_k\, v_i\, \boldsymbol{g}_j \qquad \text{etc.} \tag{25}$$

The exchange of two indices changes the sequence and hence results in a change of sign:

$$\epsilon_{ijk}\, u_i\, v_j\, \boldsymbol{g}_k = -\epsilon_{jik}\, u_i\, v_j\, \boldsymbol{g}_k = -\epsilon_{kji}\, u_i\, v_j\, \boldsymbol{g}_k. \tag{26}$$

In particular, it follows that the cross product is not commutative:

$$(\boldsymbol{u} \times \boldsymbol{v})_i = \epsilon_{ijk}\, u_j\, v_k = -\epsilon_{ikj}\, u_j\, v_k = -\epsilon_{ijk}\, u_k\, v_j = -\epsilon_{ijk}\, v_j\, u_k = -(\boldsymbol{v} \times \boldsymbol{u})_i \tag{27}$$

and therefore

$$\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u}. \tag{28}$$

The absolute value of the cross product is

$$\begin{aligned} |\boldsymbol{u} \times \boldsymbol{v}| &= |\epsilon_{ijk}\, u_i\, v_j\, \boldsymbol{g}_k| = \sqrt{\epsilon_{ijk}\, u_i\, v_j\, \boldsymbol{g}_k \cdot \epsilon_{mnl}\, u_m\, v_n\, \boldsymbol{g}_l} \\ &= \sqrt{\epsilon_{ijk}\, \epsilon_{mnl}\, u_i\, v_j\, u_m\, v_n\, \delta_{kl}} = \sqrt{\epsilon_{ijk}\, \epsilon_{mnk}\, u_i\, v_j\, u_m\, v_n} \end{aligned} \tag{29}$$

One can show that (see Problem ?)

$$\epsilon_{ijk}\, \epsilon_{imn} = \delta_{jm}\, \delta_{kn} - \delta_{jn}\, \delta_{mk}, \tag{30}$$

which is equivalent to

$$\epsilon_{ijk}\, \epsilon_{mnk} = \delta_{im}\, \delta_{jn} - \delta_{in}\, \delta_{mj}. \tag{31}$$

This allows us to write

$$\begin{aligned} |\boldsymbol{u} \times \boldsymbol{v}| &= \sqrt{(\delta_{im}\, \delta_{jn} - \delta_{in}\, \delta_{mj})\, u_i\, v_j\, u_m\, v_n} = \sqrt{(u_i\, v_j\, u_i\, v_j - u_i\, v_j\, u_j\, v_i)} \\ &= \sqrt{[u^2\, v^2 - (\boldsymbol{u} \cdot \boldsymbol{v})^2]} = \sqrt{u^2\, v^2\, [1 - \cos^2 \varphi(\boldsymbol{u}, \boldsymbol{v})]} = u\, v\, |\sin \varphi(\boldsymbol{u}, \boldsymbol{v})|, \end{aligned} \tag{32}$$

which confirms our initial definition of the cross product, cf. (15). In addition (try to show it yourself), the absolute value of the triple product $|\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})|$ equals the volume of the parallelepiped spanned by $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

0.4 Mappings

Let $\mathcal{U}, \mathcal{V} \subset \mathbb{R}^d$ be two sets and define a *mapping* $\boldsymbol{T}$ as a rule to assign to each point $\boldsymbol{u} \in \mathcal{U}$ a unique point $\boldsymbol{v} = \boldsymbol{T}(\boldsymbol{u}) \in \mathcal{V}$. We write

$$\boldsymbol{T} : \mathcal{U} \to \mathcal{V}, \quad \boldsymbol{u} \to \boldsymbol{v} = f(\boldsymbol{u}) \in \mathcal{V}, \tag{33}$$

where $\mathcal{U}$ and $\mathcal{V}$ are called the domain and the range of $\boldsymbol{T}$, respectively. Furthermore, a mapping $\boldsymbol{T} : \mathbb{R}^d \to \mathbb{R}^d$ is *linear* if and only if

$$\boldsymbol{T}(\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{T}(\boldsymbol{u}) + \boldsymbol{T}(\boldsymbol{v}) \qquad \text{and} \qquad \boldsymbol{T}(\alpha\, \boldsymbol{u}) = \alpha\, \boldsymbol{T}(\boldsymbol{u}) \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d, \quad \alpha \in \mathbb{R}. \tag{34}$$

A linear transformation $\boldsymbol{T}$ which maps vectors onto vectors is called a *second-order tensor* (one often omits the "second-order" and simply refers to a tensor). To abbreviate notation, let us write $\boldsymbol{T} \in L(\mathcal{U}, \mathcal{V})$ when expressing that $\boldsymbol{T}$ is a linear mapping of vectors in $\mathcal{U}$ onto vectors in $\mathcal{V}$.

In the following, let us understand what a tensor is. When a tensor acts on a vector $\boldsymbol{v}$, we can decompose $\boldsymbol{v}$ into its basis representation and make use of the above relations of linearity to see that

$$\boldsymbol{T}(\boldsymbol{v}) = \boldsymbol{T}(v_i\, \boldsymbol{g}_i) = v_i\, \boldsymbol{T}(\boldsymbol{g}_i), \tag{35}$$

where $\boldsymbol{T}(\boldsymbol{g}_i) \in \mathbb{R}^d$ is a vector. As a consequence, the mapping of a vector $\boldsymbol{v}$ by $\boldsymbol{T}$ is completely defined in terms of the components of $\boldsymbol{v}$ and the vectors obtained by applying the mapping $\boldsymbol{T}$

to the base vectors $\boldsymbol{g}_i$ $(i = 1, \ldots, d)$. Let us decompose these mapped base vectors into their components; for example, $\boldsymbol{T}(\boldsymbol{g}_1)$ denotes a vector whose components we call T_{11}, T_{21}, T_{31}, so that

$$\boldsymbol{T}(\boldsymbol{g}_1) = T_{11}\,\boldsymbol{g}_1 + T_{21}\,\boldsymbol{g}_2 + T_{31}\,\boldsymbol{g}_3. \tag{36}$$

Consequently, for every base vector we have

$$\boldsymbol{T}(\boldsymbol{g}_j) = T_{ij}\,\boldsymbol{g}_i. \tag{37}$$

Now, multiply this mapped vector by base vector $\boldsymbol{g}_k$ to obtain

$$\boldsymbol{T}(\boldsymbol{g}_j)\cdot\boldsymbol{g}_k = T_{ij}\,\boldsymbol{g}_i\cdot\boldsymbol{g}_k = T_{ij}\,\delta_{ik} = T_{kj}. \tag{38}$$

In summary, T_{ij} denotes the ith coordinate of the vector obtained by application of the mapping $\boldsymbol{T}$ to base vector $\boldsymbol{g}_j$. This defines the *components* of a second-order tensor. Note that, since a tensor maps vectors onto vectors, the very same principles holds that we introduced above for vector quantities: while a tensor (in symbolic notation) exists independently of the frame of reference, its components are bound to a specific choice of the coordinate system.

Let us return to the above example and apply mapping $\boldsymbol{T}$ to vector $\boldsymbol{v}$ which yields a vector $\boldsymbol{w}$ defined by

$$\boldsymbol{w} = w_j\,\boldsymbol{g}_j = \boldsymbol{T}(\boldsymbol{v}) = \boldsymbol{T}(v_i\,\boldsymbol{g}_i) = v_i\,\boldsymbol{T}(\boldsymbol{g}_i) = v_i\,T_{ji}\,\boldsymbol{g}_j \qquad\Rightarrow\qquad w_j = T_{ji}\,v_i. \tag{39}$$

The final statement (which defines the action of a second-order tensor on a vector) can easily be interpreted using matrix notation. To this end, we write the last above relation in terms of its components as

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \tag{40}$$

i.e., the action of a second-order tensor $\boldsymbol{T}$ on vector $\boldsymbol{v}$ is technically a multiplication of the matrix of the components of $\boldsymbol{T}$ by the components of vector $\boldsymbol{v}$.

Two special tensors can be defined to denote the identity mapping and the zero mapping, which are assigned special symbols:

$$\boldsymbol{T}(\boldsymbol{v}) = \boldsymbol{v}, \qquad \text{identity:}\quad \boldsymbol{T} = \boldsymbol{I} \quad\text{or}\quad \boldsymbol{T} = \mathrm{id}, \tag{41}$$

$$\boldsymbol{T}(\boldsymbol{v}) = \boldsymbol{0}, \qquad \text{zero tensor:}\quad \boldsymbol{T} = \boldsymbol{0}. \tag{42}$$

Finally, let us turn to the question of how to create a second-order tensor, or in other words: what is a second-order tensor? Using our above relations, we see by some rearrangement that

$$\boldsymbol{T}(\boldsymbol{v}) = T_{ij}\,v_j\,\boldsymbol{g}_i = T_{ij}\,\delta_{jk}\,v_k\,\boldsymbol{g}_i = T_{ij}\,(\boldsymbol{g}_j\cdot\boldsymbol{g}_k)\,v_k\,\boldsymbol{g}_i = T_{ij}\,\boldsymbol{g}_i\,(\boldsymbol{g}_j\cdot v_k\boldsymbol{g}_k) = T_{ij}\,\boldsymbol{g}_i\,(\boldsymbol{g}_j\cdot\boldsymbol{v}). \tag{43}$$

This final expression suggests to think of a second-order tensor as a quantity of the type

$$\boldsymbol{T} = T_{ij}\,\boldsymbol{g}_i\,\boldsymbol{g}_j \tag{44}$$

with the special rule

$$\boldsymbol{T}(\boldsymbol{v}) = \boldsymbol{T}\boldsymbol{v} = T_{ij}\,\boldsymbol{g}_i(\boldsymbol{g}_j\cdot\boldsymbol{v}). \tag{45}$$

Equation (44) will help to lay the basis for a more rigorous definition of second-order tensors, but first we need to find a clean interpretation of expressions of the type $\boldsymbol{g}_i\,\boldsymbol{g}_j$. While many authors use this notation, we will find it inconvenient for many reasons (simply writing two vectors next to each other is not always unambiguous). Instead, by convention we insert the symbol $\otimes$ between any two quantities that are not meant to be multiplied by any of the multiplication rules we defined above. For example, a multiplication of a vector by a scalar, $\alpha\,\boldsymbol{v}$, could be written $\alpha \otimes \boldsymbol{v}$ but this is not common because $\alpha\,\boldsymbol{v}$ is unambiguous. The above definition of a second-order tensor may be written as $\boldsymbol{T} = T_{ij}\,\boldsymbol{g}_i \otimes \boldsymbol{g}_j$. With this convention, we are now in place to introduce the outer product of two vectors.

0.5 Outer Product of Two Vectors

The outer (or dyadic) product of two vectors is often termed the *tensor product* because it results in a tensor quantity of second order. The outer product of vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^d$ is formally denoted by

$$\boldsymbol{v} \otimes \boldsymbol{w} \tag{46}$$

and is defined by the relation (with $\boldsymbol{u} \in \mathbb{R}^d$)

$$(\boldsymbol{v} \otimes \boldsymbol{w})\boldsymbol{u} = \boldsymbol{v}\,(\boldsymbol{w} \cdot \boldsymbol{u}), \tag{47}$$

i.e., the action of tensor $(\boldsymbol{v} \otimes \boldsymbol{w})$ on vector $\boldsymbol{u}$ is defined through the above relation. Hence, the tensor product defines a mapping of vectors $\boldsymbol{u}$ onto the direction of $\boldsymbol{v}$ (and the length of the new vector is given by a projection onto $\boldsymbol{w}$), i.e. $(\boldsymbol{v} \otimes \boldsymbol{u}) \in L(\mathbb{R}^d, \mathbb{R}^d)$. One can easily verify that the tensor $\boldsymbol{T} = \boldsymbol{v} \otimes \boldsymbol{w}$ indeed defines a linear mapping:

$$\begin{aligned}
\boldsymbol{T}(\alpha\,\boldsymbol{u} + \beta\,\boldsymbol{v}) = (\boldsymbol{v} \otimes \boldsymbol{w})(\alpha\,\boldsymbol{u} + \beta\,\boldsymbol{v}) &= \boldsymbol{v}\ [\boldsymbol{w} \cdot (\alpha\,\boldsymbol{u} + \beta\,\boldsymbol{v})] && (48)\\
&= \boldsymbol{v}\ [\alpha\,\boldsymbol{w} \cdot \boldsymbol{u} + \beta\,\boldsymbol{w} \cdot \boldsymbol{v}] = \alpha\,\boldsymbol{v}(\boldsymbol{w} \cdot \boldsymbol{u}) + \beta\,\boldsymbol{v}(\boldsymbol{w} \cdot \boldsymbol{v}) && (49)\\
&= \alpha\,(\boldsymbol{v} \otimes \boldsymbol{w})\boldsymbol{u} + \beta\,(\boldsymbol{v} \otimes \boldsymbol{w})\boldsymbol{v} = \alpha\,\boldsymbol{T}\boldsymbol{u} + \beta\,\boldsymbol{T}\boldsymbol{v}. && (50)
\end{aligned}$$

Using index notation, the outer product of two tensors is written in the following form:

$$\boldsymbol{T} = \boldsymbol{v} \otimes \boldsymbol{w} = v_i\,\boldsymbol{g}_i \otimes w_j\,\boldsymbol{g}_j = v_i\,w_j\,\boldsymbol{g}_i \otimes \boldsymbol{g}_j = T_{ij}\,\boldsymbol{g}_i \otimes \boldsymbol{g}_j \tag{51}$$

and one can identify the components of tensor $\boldsymbol{T}$ as $T_{ij} = v_i\,w_j$, i.e. we have

$$[\boldsymbol{v} \otimes \boldsymbol{w}] = [v_i\,w_j] = \begin{bmatrix} v_1\,w_1 & v_1\,w_2 & v_1\,w_3 \\ v_2\,w_1 & v_2\,w_2 & v_2\,w_3 \\ v_3\,w_1 & v_3\,w_2 & v_3\,w_3 \end{bmatrix}. \tag{52}$$

Now, we can apply a second-order tensor $\boldsymbol{T}$ to an arbitrary vector $\boldsymbol{u}$, using definition (47):

$$\boldsymbol{T}\boldsymbol{u} = (T_{ij}\,\boldsymbol{g}_i \otimes \boldsymbol{g}_j)\,(u_k\,\boldsymbol{g}_k) = T_{ij}\,\boldsymbol{g}_i\,(\boldsymbol{g}_j \cdot u_k\boldsymbol{g}_k) = T_{ij}\,u_k\,\boldsymbol{g}_i\,\delta_{jk} = T_{ij}\,u_j\,\boldsymbol{g}_i \tag{53}$$

and, of course, the result is a vector quantity with components $(\boldsymbol{T}\boldsymbol{u})_i = T_{ij}u_j$. Note that various authors denote the action of a second-order tensor $\boldsymbol{T}$ on a vector $\boldsymbol{u}$ by $\boldsymbol{T} \cdot \boldsymbol{u}$. Here, however, we reserve the $\cdot$ symbol for scalar products, and we use the notation $\boldsymbol{T}\boldsymbol{u}$ for tensor-vector operations as introduced above. As mentioned before, when using index notation in practice one usually omits the base vectors entirely, so that from now on we may write, e.g., the equation $\boldsymbol{w} = \boldsymbol{T}\boldsymbol{u}$ as $w_i = T_{ij}\, u_j$, where the only free index i indicates that this represents a vector equation.

Vector $\boldsymbol{T}\boldsymbol{u}$ can further be multiplied by another vector $\boldsymbol{v} \in \mathbb{R}^d$, resulting in a scalar quantity:

$$\boldsymbol{v} \cdot \boldsymbol{T}\boldsymbol{u} = v_k\, \boldsymbol{g}_k \cdot T_{ij}\, u_j\, \boldsymbol{g}_i = v_i\, T_{ij}\, u_j. \tag{54}$$

Consequently, the components of a tensor of second order can easily be obtained from a multiplication by the corresponding basis vectors (in an orthonormal basis) in agreement with (38):

$$T_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{T}\boldsymbol{g}_j = \boldsymbol{T}(\boldsymbol{g}_j) \cdot \boldsymbol{g}_i. \tag{55}$$

We can extend the above definition of a second-order tensor to general tensors of arbitrary order by expressing a *tensor of order* n by

$$\boldsymbol{T} = T_{ij\ldots p}\, \boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \ldots \otimes \boldsymbol{g}_p, \tag{56}$$

where we must have n indices and n base vectors connected by outer products. Consequently, a scalar quantity is referred to as a *tensor of zeroth order*. A vector quantity can be viewed as a *tensor of first order*, and quite naturally we recover our definition of tensors of second order. Tensors of order two and higher will usually be denoted by upper case bold letters.

Finally, note that the above concept is not limited to the outer product of vectors only, but we can easily extend this concept to general tensors of arbitrary order by defining

$$\begin{aligned}\boldsymbol{T} \otimes \boldsymbol{S} &= T_{ij\ldots p}\, \boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \ldots \otimes \boldsymbol{g}_p \otimes S_{ab\ldots d}\, \boldsymbol{g}_a \otimes \boldsymbol{g}_b \otimes \ldots \otimes \boldsymbol{g}_d \\ &= T_{ij\ldots p}\, S_{ab\ldots d}\, \boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \ldots \otimes \boldsymbol{g}_p \otimes \boldsymbol{g}_a \otimes \boldsymbol{g}_b \otimes \ldots \otimes \boldsymbol{g}_d.\end{aligned}$$

For subsequent sections, keep in mind relation (47) and its generalization to the cross product, respectively,

$$(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w} = \boldsymbol{u}\,(\boldsymbol{v} \cdot \boldsymbol{w}) \qquad \text{and} \qquad (\boldsymbol{u} \otimes \boldsymbol{v}) \times \boldsymbol{w} = \boldsymbol{u}\,(\boldsymbol{v} \times \boldsymbol{w}). \tag{57}$$

In addition, from our definition of the outer product follow a number of relations which will be helpful in everyday tensor life (with $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^d$, $\alpha \in \mathbb{R}$):

$$\begin{aligned}&(\boldsymbol{v} + \boldsymbol{w}) \otimes \boldsymbol{u} = \boldsymbol{v} \otimes \boldsymbol{u} + \boldsymbol{w} \otimes \boldsymbol{u} \\ &\boldsymbol{u} \otimes (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \otimes \boldsymbol{v} + \boldsymbol{u} \otimes \boldsymbol{w} \\ &\alpha\, \boldsymbol{a} \otimes \boldsymbol{b} = (\alpha\, \boldsymbol{a}) \otimes \boldsymbol{b} = \boldsymbol{a} \otimes (\alpha \boldsymbol{b}) \\ &\alpha\, \boldsymbol{u}(\boldsymbol{v} \cdot \boldsymbol{w}) = \alpha\, (\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w} \\ &\alpha\, \boldsymbol{u} \cdot (\boldsymbol{v} \otimes \boldsymbol{w}) = \alpha\, (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{w}\end{aligned}$$

0.6 Transpose, Symmetric and Asymmetric Tensors, Trace

The transpose of a tensor $\boldsymbol{T}$ is denoted by $\boldsymbol{T}^\mathsf{T}$ and defined through the relation

$$\boldsymbol{u} \cdot \boldsymbol{T}\boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{T}^\mathsf{T}\boldsymbol{u} \qquad \Leftrightarrow \qquad u_i\, T_{ij} v_j = v_i\, T_{ij} u_j = v_j\, T_{ji} u_i = u_i\, T_{ji} v_j. \tag{58}$$

Therefore,

$$\boldsymbol{T}^\mathsf{T} = T_{ji}\, \boldsymbol{g}_i \otimes \boldsymbol{g}_j \qquad \text{or} \qquad (\boldsymbol{T}^\mathsf{T})_{ij} = T_{ji} \tag{59}$$

(i.e., transposition simply exchanges the two indices of a second-order tensor). As a consequence, we see that $(\boldsymbol{T}^\mathsf{T})^\mathsf{T} = \boldsymbol{T}$. Furthermore, we conclude that

$$(\boldsymbol{u} \otimes \boldsymbol{v})^\mathsf{T} = (u_i\, v_j\, \boldsymbol{g}_i \otimes \boldsymbol{g}_j)^\mathsf{T} = u_j\, v_i\, \boldsymbol{g}_i \otimes \boldsymbol{g}_j = v_i\, \boldsymbol{g}_i \otimes u_j\, \boldsymbol{g}_j = \boldsymbol{v} \otimes \boldsymbol{u}. \tag{60}$$

A tensor is called

$$\textit{symmetric} \text{ if } \quad \boldsymbol{T} = \boldsymbol{T}^\mathsf{T} \quad \text{or} \quad T_{ij} = T_{ji} \tag{61}$$

$$\textit{skew-symmetric} \text{ or } \textit{antimetric} \text{ if } \quad \boldsymbol{T} = -\boldsymbol{T}^\mathsf{T} \quad \text{or} \quad T_{ij} = -T_{ji}. \tag{62}$$

For a skew-symmetric tensor, we have $T_{ij} = -T_{ij}$ and hence it follows that $T_{(ii)} = -T_{(ii)} = 0$, i.e., the diagonal components of a skew-symmetric tensor vanish.

In addition, we define the symmetric and skew-symmetric parts of a tensor $\boldsymbol{T}$ by, respectively,

$$\operatorname{sym} \boldsymbol{T} = \tfrac{1}{2}(\boldsymbol{T} + \boldsymbol{T}^\mathsf{T}), \qquad \operatorname{skew} \boldsymbol{T} = \tfrac{1}{2}(\boldsymbol{T} - \boldsymbol{T}^\mathsf{T}) \tag{63}$$

so that

$$\boldsymbol{T} = \operatorname{sym} \boldsymbol{T} + \operatorname{skew} \boldsymbol{T}. \tag{64}$$

The *trace* of a tensor is a linear operation which yields a scalar property of $\boldsymbol{T}$ defined by

$$\operatorname{tr}(\boldsymbol{u} \otimes \boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v} \qquad \Rightarrow \qquad \operatorname{tr} \boldsymbol{T} = \operatorname{tr}(T_{ij}\, \boldsymbol{g}_i \otimes \boldsymbol{g}_j) = T_{ij}\, \boldsymbol{g}_i \cdot \boldsymbol{g}_j = T_{ii}, \tag{65}$$

i.e., the sum of all diagonal entries. This quantity is invariant under coordinate rotations as we will see later. As we discussed before, the components of vectors and tensors change when moving to a different basis, the vector or tensor quantity itself, however, does not change (written in symbolic notation). Here, we encounter a first example of a property of a tensor that consists of components yet does not change under coordinate changes: the trace. We will see other such properties later.

0.7 Composite Mapping, Multiplication of Tensors

Let us apply two mappings $\boldsymbol{S}$ and $\boldsymbol{T}$ to a vector $\boldsymbol{u}$ in the following order: first map $\boldsymbol{u}$ by application of $\boldsymbol{T}$ to obtain $\boldsymbol{T}\boldsymbol{u}$, and then apply mapping $\boldsymbol{S}$. In summary, we apply the composite mapping $\boldsymbol{S} \circ \boldsymbol{T}$ which can be written as

$$\boldsymbol{S}(\boldsymbol{T}\boldsymbol{u}) = (\boldsymbol{S} \circ \boldsymbol{T})\boldsymbol{u}. \tag{66}$$

In a straightforward manner, following our above introduction of a tensor quantity, one can show that the composite mapping is nothing but a multiplication of the components of the two tensors,

$$\boldsymbol{S}(\boldsymbol{T}\boldsymbol{u}) = (\boldsymbol{S}\boldsymbol{T})\boldsymbol{u} = \boldsymbol{S}\boldsymbol{T}\boldsymbol{u}, \tag{67}$$

and in indicial notation we have

$$\boldsymbol{S}(\boldsymbol{T}\boldsymbol{u}) = \boldsymbol{S}(T_{ij}\, u_j\, \boldsymbol{g}_i) = S_{kl}\, T_{ij}\, u_j\, \boldsymbol{g}_k\, (\boldsymbol{g}_l \cdot \boldsymbol{g}_i) = S_{ki}\, T_{ij}\, u_j\, \boldsymbol{g}_k \tag{68}$$

$$= (S_{ki}\, T_{il})\, \boldsymbol{g}_k \otimes \boldsymbol{g}_l \cdot u_j\, \boldsymbol{g}_j = (\boldsymbol{S}\boldsymbol{T})\boldsymbol{u} \tag{69}$$

so that it makes sense to introduce the multiplication of two tensors as (renaming indices for simplicity)

$$\boldsymbol{S}\boldsymbol{T} = S_{ik}\,T_{kj}\,\boldsymbol{g}_i \otimes \boldsymbol{g}_j \qquad \Leftrightarrow \qquad (\boldsymbol{S}\boldsymbol{T})_{ij} = S_{ik}\,T_{kj}. \tag{70}$$

This defines the multiplication of two second-order tensors. In addition, as a logical consequence of writing the above statement out in full base vector representation, we arrive at the general rule

$$(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{u} \otimes \boldsymbol{v}) = \boldsymbol{a} \otimes (\boldsymbol{b} \cdot \boldsymbol{u})\boldsymbol{v} = (\boldsymbol{b} \cdot \boldsymbol{u})\,\boldsymbol{a} \otimes \boldsymbol{v} \tag{71}$$

so that

$$\begin{aligned}\boldsymbol{S}\boldsymbol{T} &= (S_{ij}\,\boldsymbol{g}_i \otimes \boldsymbol{g}_j)(T_{kl}\,\boldsymbol{g}_k \otimes \boldsymbol{g}_l) = S_{ij}\,T_{kl}\,\boldsymbol{g}_i \otimes (\boldsymbol{g}_j \cdot \boldsymbol{g}_k)\,\boldsymbol{g}_l \\ &= S_{ij}\,T_{kl}\,\delta_{jk}\,\boldsymbol{g}_i \otimes \boldsymbol{g}_l = S_{ik}\,T_{kl}\,\boldsymbol{g}_i \otimes \boldsymbol{g}_l\end{aligned} \tag{72}$$

which agrees with the above definition. Conversely, one obtains that

$$(\boldsymbol{S}\boldsymbol{T}^\mathsf{T})_{ij} = S_{ik}\,T_{jk} \qquad \text{and} \qquad (\boldsymbol{S}^\mathsf{T}\boldsymbol{T})_{ij} = S_{ki}\,T_{kj}. \tag{73}$$

A further important relation defines the transpose of a composite mapping:

$$(\boldsymbol{S}\boldsymbol{T})^\mathsf{T} = (S_{ik}\,T_{kl}\,\boldsymbol{g}_i \otimes \boldsymbol{g}_l)^\mathsf{T} = S_{lk}\,T_{ki}\,\boldsymbol{g}_i \otimes \boldsymbol{g}_l = T_{ki}\,S_{ln}\,(\boldsymbol{g}_i \otimes \boldsymbol{g}_k)(\boldsymbol{g}_n \otimes \boldsymbol{g}_l) = \boldsymbol{T}^\mathsf{T}\boldsymbol{S}^\mathsf{T}. \tag{74}$$

Finally, we can make use of our above concept to derive general tensor multiplication rules such as, e.g., the following:

$$\begin{aligned}(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{T} &= u_i\,v_j(\boldsymbol{g}_i \otimes \boldsymbol{g}_j)\,T_{kl}(\boldsymbol{g}_k \otimes \boldsymbol{g}_l) = u_i\,v_j\,T_{kl}(\boldsymbol{g}_i \otimes \boldsymbol{g}_j)(\boldsymbol{g}_k \otimes \boldsymbol{g}_l) \\ &= u_i\,v_j\,T_{kl}(\boldsymbol{g}_j \cdot \boldsymbol{g}_k)(\boldsymbol{g}_i \otimes \boldsymbol{g}_l) = u_i\,v_j\,T_{jl}\,\boldsymbol{g}_i \otimes \boldsymbol{g}_l = u_i\,\boldsymbol{g}_i \otimes T_{jl}\,v_j\boldsymbol{g}_l = \boldsymbol{u} \otimes \boldsymbol{T}^\mathsf{T}\boldsymbol{v},\end{aligned} \tag{75}$$

which implies

$$(\boldsymbol{u} \otimes \boldsymbol{v})\,\boldsymbol{T} = \boldsymbol{u} \otimes \boldsymbol{T}^\mathsf{T}\boldsymbol{v}. \tag{76}$$

This could have been expected: using the rule (74) we see that

$$[(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{T}]^\mathsf{T} = \boldsymbol{T}^\mathsf{T}\,(\boldsymbol{u} \otimes \boldsymbol{v})^\mathsf{T} = \boldsymbol{T}^\mathsf{T}\,(\boldsymbol{v} \otimes \boldsymbol{u}) = (\boldsymbol{T}^\mathsf{T}\,\boldsymbol{v}) \otimes \boldsymbol{u}. \tag{77}$$

If we now use the fact that $(\boldsymbol{a} \otimes \boldsymbol{b})^\mathsf{T} = \boldsymbol{b} \otimes \boldsymbol{a}$, we obtain in accordance with (75)

$$(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{T} = \left[(\boldsymbol{T}^\mathsf{T}\,\boldsymbol{v}) \otimes \boldsymbol{u}\right]^\mathsf{T} = \boldsymbol{u} \otimes \boldsymbol{T}^\mathsf{T}\,\boldsymbol{v}. \tag{78}$$

Note that some authors make use of the equivalent notations $\boldsymbol{T}^\mathsf{T}\boldsymbol{v} = \boldsymbol{v}^\mathsf{T}\boldsymbol{T} = \boldsymbol{v} \cdot \boldsymbol{T}$ (all of which meaning the same), so that one could write equivalently, e.g., $(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{T} = \boldsymbol{u} \otimes (\boldsymbol{v}^\mathsf{T}\boldsymbol{T})$. We will, however, avoid this notation to avoid ambiguity.

As an example, consider the trace of the composite mapping $\boldsymbol{S}\boldsymbol{T}$ where $\boldsymbol{T} = \boldsymbol{u} \otimes \boldsymbol{v}$:

$$\boldsymbol{S}\boldsymbol{T} = \boldsymbol{S}(\boldsymbol{u}\otimes\boldsymbol{v}) = S_{ij}\,u_j\,v_k\,\boldsymbol{g}_i\otimes\boldsymbol{g}_k \qquad \Rightarrow \qquad \operatorname{tr}(\boldsymbol{S}\boldsymbol{T}) = S_{ij}\,u_j\,v_k\,\delta_{ik} = S_{ij}\,u_j\,v_i = \boldsymbol{v}\cdot\boldsymbol{S}\,\boldsymbol{u}. \tag{79}$$

Next, assume the above composite mapping acts on a vector $\boldsymbol{w}$, resulting in vector $\boldsymbol{z}$, i.e.

$$\boldsymbol{z} = \boldsymbol{S}(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w}. \tag{80}$$

Let us formulate this vector equation using indicial notation:

$$\boldsymbol{z} = z_i\,\boldsymbol{g}_i = \boldsymbol{S}(\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{w} = (\boldsymbol{S}\,\boldsymbol{u})(\boldsymbol{v} \cdot \boldsymbol{w}) = S_{ij}\,u_j\,\boldsymbol{g}_i\,(v_k\,w_k) = S_{ij}\,u_j\,v_k\,w_k\,\boldsymbol{g}_i. \tag{81}$$

Therefore, the same statement in inidicial notation reads

$$z_i = S_{ij}\,u_j\,v_k\,w_k. \tag{82}$$

0.8 Inner Product of Two Tensors, Norm of a Tensor

Having introduced tensors of higher-orders, let us now extend our definition of the inner product (which we introduced for two vectors) to tensors of arbitrary order. To this end, we define the inner product of two general tensors of the same order via inner products of all pairs of base vectors, i.e.,

$$\begin{aligned}\boldsymbol{S}\cdot\boldsymbol{T} &= (S_{ij\dots n}\,\boldsymbol{g}_i\otimes\boldsymbol{g}_j\otimes\dots\otimes\boldsymbol{g}_n)\cdot(T_{kl\dots m}\,\boldsymbol{g}_k\otimes\boldsymbol{g}_l\otimes\dots\otimes\boldsymbol{g}_m)\\ &= S_{ij\dots n}T_{kl\dots m}(\boldsymbol{g}_i\cdot\boldsymbol{g}_k)(\boldsymbol{g}_j\cdot\boldsymbol{g}_l)\dots(\boldsymbol{g}_n\cdot\boldsymbol{g}_m) = S_{ij\dots n}T_{ij\dots n}.\end{aligned}$$

As a consequence, we recover the inner product of two vectors as

$$\boldsymbol{v}\cdot\boldsymbol{u} = v_i\,u_j(\boldsymbol{g}_i\cdot\boldsymbol{g}_j) = v_i\,u_i. \tag{83}$$

The inner product of two tensors of second-order becomes

$$\boldsymbol{S}\cdot\boldsymbol{T} = S_{ij}\,T_{ij}, \tag{84}$$

which can alternatively be defined in the following fashion:

$$\boldsymbol{S}\cdot\boldsymbol{T} = \operatorname{tr}(\boldsymbol{S}\boldsymbol{T}^\mathsf{T}) = \operatorname{tr}(\boldsymbol{S}^\mathsf{T}\boldsymbol{T}) = S_{ij}\,T_{ij}. \tag{85}$$

Note that this inner product of two tensors is only defined for two tensors *of the same order*, and it always results in a scalar quantity, which is why it is often referred to as the *scalar product.* Furthermore, note that this notation varies from author to author even within the continuum mechanics community (some authors denote our $\boldsymbol{T}\boldsymbol{u}$ by $\boldsymbol{T}\cdot\boldsymbol{u}$ and our $\boldsymbol{S}\cdot\boldsymbol{T}$ by $\boldsymbol{S}:\boldsymbol{T}$). Here, we will exclusively use the notation introduced above. Besides, when using indicial notation no such ambiguity exists.

In analogy to the norm of a vector, we define the *norm of a tensor* (the so-called Hilbert-Schmidt norm) by

$$|\boldsymbol{T}| = \sqrt{\boldsymbol{T}\cdot\boldsymbol{T}} = \sqrt{T_{ij}\,T_{ij}}. \tag{86}$$

0.9 Determinant of a Tensor

Next, let us introduce the determinant of a tensor, which is defined by

$$\det\boldsymbol{T} = \epsilon_{ijk}\,T_{i1}\,T_{j2}\,T_{k3} = \epsilon_{ijk}\,T_{1i}\,T_{2j}\,T_{3k}, \tag{87}$$

from which it follows directly that $\det\boldsymbol{T}^\mathsf{T} = \det\boldsymbol{T}$. A rearrangement of indices shows that

$$\det(\boldsymbol{S}\boldsymbol{T}) = \det\boldsymbol{S}\,\det\boldsymbol{T}. \tag{88}$$

0.10 Inverse of a Tensor

If a mapping $\boldsymbol{T}\in L(\mathbb{R}^d,\mathbb{R}^d)$ is bijective (i.e., to every mapped point $\boldsymbol{w}=\boldsymbol{T}\boldsymbol{v}$ there corresponds one unique point $\boldsymbol{v}$) we say that tensor $\boldsymbol{T}$ is invertible. This is equivalent to requiring that for every $\boldsymbol{w} \in \mathbb{R}^d$ there exist a $\boldsymbol{v} \in \mathbb{R}^d$ that solves the equation $\boldsymbol{T}\boldsymbol{v} = \boldsymbol{w}$. This solution is then written as

$$\boldsymbol{v} = \boldsymbol{T}^{-1}(\boldsymbol{w}) = \boldsymbol{T}^{-1}\boldsymbol{w} \qquad \text{or} \qquad v_i = T_{ij}^{-1}\, w_j, \tag{89}$$

where $\boldsymbol{T}^{-1}$ is the *inverse* mapping of $\boldsymbol{T}$. As we will see later, a tensor $\boldsymbol{T}$ is invertible if $\det \boldsymbol{T} \neq 0$.

If we first apply an invertible mapping $\boldsymbol{T}$ to a vector $\boldsymbol{u}$ and then its inverse $\boldsymbol{T}^{-1}$, we must recover the original vector (and the same must hold true in reversed order), i.e.,

$$\boldsymbol{T}^{-1}(\boldsymbol{T}\boldsymbol{u}) = \boldsymbol{T}^{-1}\boldsymbol{T}\boldsymbol{u} = \boldsymbol{u}, \qquad \boldsymbol{T}(\boldsymbol{T}^{-1}\boldsymbol{u}) = \boldsymbol{T}\boldsymbol{T}^{-1}\boldsymbol{u} = \boldsymbol{u} \tag{90}$$

and therefore we must have

$$\boldsymbol{T}\boldsymbol{T}^{-1} = \boldsymbol{T}^{-1}\boldsymbol{T} = \boldsymbol{I}. \tag{91}$$

A tensor is called *orthogonal* if $\boldsymbol{T}^{-1} = \boldsymbol{T}^\mathsf{T}$ such that $\boldsymbol{T}\boldsymbol{T}^\mathsf{T} = \boldsymbol{T}^\mathsf{T}\boldsymbol{T} = \boldsymbol{I}$. Consequently, by taking the determinant on both sides of that equation, we see that all orthogonal tensors $\boldsymbol{T} \in L(\mathbb{R}^d, \mathbb{R}^d)$ satisfy

$$\det(\boldsymbol{T}\boldsymbol{T}^\mathsf{T}) = \det \boldsymbol{T} \cdot \det \boldsymbol{T}^\mathsf{T} = (\det \boldsymbol{T})^2 = 1 \qquad \Rightarrow \qquad \det \boldsymbol{T} = \pm 1. \tag{92}$$

All such orthogonal tensors form the *orthogonal group* of the d-dimensional space, abbreviated by $O(d)$. If we enforce $\det \boldsymbol{T} = 1$, then we arrive at the *special orthogonal group* $SO(d)$:

$$\begin{aligned} \boldsymbol{T}\boldsymbol{T}^\mathsf{T} = \boldsymbol{T}^\mathsf{T}\boldsymbol{T} = \boldsymbol{I} \qquad &\Leftrightarrow \qquad \boldsymbol{T} \in O(d), \\ \boldsymbol{T}\boldsymbol{T}^\mathsf{T} = \boldsymbol{T}^\mathsf{T}\boldsymbol{T} = \boldsymbol{I} \quad \text{and} \quad \det \boldsymbol{T} = 1 \qquad &\Leftrightarrow \qquad \boldsymbol{T} \in SO(d). \end{aligned}$$

We will later see that orthogonal tensors with determinant $+1$ represent mappings which describe rotations in space, whereas orthogonal tensors with determinant -1 correspond to mappings which describe reflections.

Now, let us apply two mappings $\boldsymbol{T}$ and $\boldsymbol{S}$ to some vector $\boldsymbol{u}$ and subsequently apply the inverse mappings; i.e., we first apply $\boldsymbol{T}$, then $\boldsymbol{S}$, then we use the inverse $\boldsymbol{S}^{-1}$ and finally the inverse $\boldsymbol{T}^{-1}$. At the end we must recover the original $\boldsymbol{u}$, i.e.,

$$\boldsymbol{T}^{-1}(\boldsymbol{S}^{-1}(\boldsymbol{S}(\boldsymbol{T}\boldsymbol{u}))) = \boldsymbol{u}. \tag{93}$$

But we could as well define a composite mapping $(\boldsymbol{S}\boldsymbol{T})$ and first apply this mapping and then its inverse, which also results in the identity

$$(\boldsymbol{S}\boldsymbol{T})^{-1}(\boldsymbol{S}\boldsymbol{T})\boldsymbol{u} = \boldsymbol{u}. \tag{94}$$

By rearranging parentheses in (93), we have $\boldsymbol{T}^{-1}(\boldsymbol{S}^{-1}(\boldsymbol{S}(\boldsymbol{T}\boldsymbol{u}))) = (\boldsymbol{T}^{-1}\boldsymbol{S}^{-1})(\boldsymbol{S}\boldsymbol{T})\boldsymbol{u}$. A comparison with (94) reveals that

$$(\boldsymbol{S}\boldsymbol{T})^{-1} = \boldsymbol{T}^{-1}\boldsymbol{S}^{-1}. \tag{95}$$

We can establish the following relation for the determinant of an inverse:

$$\boldsymbol{T}\boldsymbol{T}^{-1} = \boldsymbol{I} \qquad \Rightarrow \qquad \det(\boldsymbol{T}\boldsymbol{T}^{-1}) = \det \boldsymbol{T}\, \det \boldsymbol{T}^{-1} = 1 \qquad \Rightarrow \qquad \det \boldsymbol{T}^{-1} = 1/\det \boldsymbol{T}. \tag{96}$$

A common notation is to combine the symbols for transposition and inversion and write for brevity

$$(\boldsymbol{T}^{-1})^\mathsf{T} = (\boldsymbol{T}^\mathsf{T})^{-1} = \boldsymbol{T}^{-T}. \tag{97}$$

0.11 About Notation

The aforementioned indicial notation is widely used within the field of continuum mechanics and a helpful tool when evaluating complex tensor operations. For many applications it is convenient to omit the base vectors and only formulate relations in terms of tensor components (whenever it is unambiguous). For example, instead of writing $\boldsymbol{v}$ we could write v_i and the single index makes clear that we refer to a vector (a tensor of first order). This notation, however, is not unique when dealing with tensors of higher order. For example, writing S_{ij} without showing the base vectors does not indicate whether we mean $\boldsymbol{S} = S_{ij}\boldsymbol{g}_i \otimes \boldsymbol{g}_j$ or $\boldsymbol{S}^\top = S_{ij}\boldsymbol{g}_j \otimes \boldsymbol{g}_i$. While this is of no importance for vector quantities that only carry one free index, it is essential when dealing with higher-order tensors. Therefore, we must ensure that our notation is unique. When writing entire equations in indicial notation, this is always guaranteed since the same indices have the same meaning on both sides of the equation. For example, instead of writing $\boldsymbol{z} = \boldsymbol{T}\boldsymbol{u}$ or $z_i\,\boldsymbol{g}_i = T_{ij}\,u_j\,\boldsymbol{g}_i$ we may write $z_i = T_{ij}u_j$ for short (notice that it is only meaningful to omit the base vectors, if they carry the same indices on both sides of the equation). We can apply the same rule to tensor equations: $S_{ij} = u_i\,v_j$ is unambiguous because it can either mean $\boldsymbol{S} = \boldsymbol{u} \otimes \boldsymbol{v}$ or $\boldsymbol{S}^\top = \boldsymbol{v} \otimes \boldsymbol{u}$, but these two equations are indentical. When writing a single tensor quantity of higher order, we should clarify what we mean, e.g., by writing $[\boldsymbol{S}]_{ij}$ which implies the omitted base vectors are $\boldsymbol{g}_i \otimes \boldsymbol{g}_j$ (in the same order as the indices given after $[\ldots]$). Table 1 summarizes some of the most important examples of using indicial notation.

Also, note that all of the above definitions of vector operations can be extended to tensors of arbitrary order by writing the tensor in its base representations and applying the vector operations directly to the base vectors, e.g., $\boldsymbol{T} \times \boldsymbol{u}$ be defined as

$$\boldsymbol{T} \times \boldsymbol{u} = T_{ij}\,\boldsymbol{g}_i \otimes \boldsymbol{g}_j \times u_k\,\boldsymbol{g}_k = T_{ij}\,u_k\,\boldsymbol{g}_i \otimes (\boldsymbol{g}_j \times \boldsymbol{g}_k) = \epsilon_{jkl}\,T_{ij}\,u_k\,\boldsymbol{g}_i \otimes \boldsymbol{g}_l. \tag{98}$$

symbolic notation	full component form	indicial notation
$\boldsymbol{v}$	$v_i\boldsymbol{g}_i$	v_i
$\boldsymbol{v}\cdot\boldsymbol{w}$	v_iw_i	v_iw_i
$\boldsymbol{v}\otimes\boldsymbol{w}$	$v_iw_j\boldsymbol{g}_i\otimes\boldsymbol{g}_j$	$[\boldsymbol{v}\otimes\boldsymbol{w}]_{ij} = v_iw_j$
$\boldsymbol{v}\times\boldsymbol{w}$	$v_iw_j\epsilon_{ijk}\boldsymbol{g}_k$	$[\boldsymbol{v}\times\boldsymbol{w}]_{ij} = v_iw_j\epsilon_{ijk}$
$\boldsymbol{v}\cdot(\boldsymbol{w}\times\boldsymbol{x})$	$v_iw_jx_k\epsilon_{ijk}$	$v_iw_jx_k\epsilon_{ijk}$
$\boldsymbol{T}$	$T_{ij}\boldsymbol{g}_i\otimes\boldsymbol{g}_j$	$[\boldsymbol{T}]_{ij} = T_{ij}$
$\boldsymbol{T}\boldsymbol{v}$	$T_{ij}v_j\boldsymbol{g}_i$	$[\boldsymbol{T}\boldsymbol{v}]_i = T_{ij}v_j$
$\boldsymbol{T}^\top\boldsymbol{v}$	$v_iT_{ij}\boldsymbol{g}_j$	$[\boldsymbol{T}^\top\boldsymbol{v}]_j = v_iT_{ij}$
$\boldsymbol{v}\cdot\boldsymbol{T}\boldsymbol{w}$	$v_iT_{ij}w_j$	$v_iT_{ij}w_j$
$\boldsymbol{S}\boldsymbol{T}$	$T_{ij}S_{jk}\boldsymbol{g}_i\otimes\boldsymbol{g}_k$	$[\boldsymbol{S}\boldsymbol{T}]_{ik} = T_{ij}S_{jk}$
$\boldsymbol{S}^\top\boldsymbol{T}$	$T_{ji}S_{jk}\boldsymbol{g}_i\otimes\boldsymbol{g}_k$	$[\boldsymbol{S}^\top\boldsymbol{T}]_{ik} = T_{ji}S_{jk}$
$\boldsymbol{S}\boldsymbol{T}^\top$	$T_{ij}S_{kj}\boldsymbol{g}_i\otimes\boldsymbol{g}_k$	$[\boldsymbol{S}\boldsymbol{T}^\top]_{ik} = T_{ij}S_{kj}$
$\boldsymbol{T}\times\boldsymbol{v}$	$T_{ij}v_k\epsilon_{jkl}\boldsymbol{g}_i\otimes\boldsymbol{g}_l$	$[\boldsymbol{T}\times\boldsymbol{v}]_{il} = T_{ij}v_k\epsilon_{jkl}$
$\boldsymbol{T}\otimes\boldsymbol{v}$	$T_{ij}v_k\boldsymbol{g}_i\otimes\boldsymbol{g}_j\otimes\boldsymbol{g}_k$	$[\boldsymbol{T}\otimes\boldsymbol{v}]_{ijk} = T_{ij}v_k$

Table 1: Overview of some basic tensor operations in symbolic and indicial notation.

1 Vector and Tensor Analysis in a Cartesian Basis

1.1 Fields, Gateaux Derivative, Gradient, Divergence, and Curl

In continuum mechanics, we assume that all variables of interest are continuous functions of position $\boldsymbol{x} \in \mathbb{R}^d$ and time $t \in \mathbb{R}$. Therefore, let us introduce such fields which may be classified into the following categories (with examples):

$$\begin{array}{llll}
\text{scalar fields:} & \phi: & \mathbb{R} \to \mathbb{R}, & t \to \phi(t) \\
 & \phi: & \mathbb{R}^d \to \mathbb{R}, & \boldsymbol{x} \to \phi(\boldsymbol{x}) \\
 & \phi: & \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}, & (\boldsymbol{x}, t) \to \phi(\boldsymbol{x}, t) \\
\text{vector fields:} & \boldsymbol{v}: & \mathbb{R} \to \mathbb{R}^d, & t \to \boldsymbol{v}(t) \\
 & \boldsymbol{v}: & \mathbb{R}^d \to \mathbb{R}^d, & \boldsymbol{x} \to \boldsymbol{v}(\boldsymbol{x}) \\
 & \boldsymbol{v}: & \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d, & (\boldsymbol{x}, t) \to \boldsymbol{v}(\boldsymbol{x}, t) \\
\text{tensor fields:} & \boldsymbol{T}: & \mathbb{R} \to L(\mathbb{R}^d, \mathbb{R}^d), & t \to \boldsymbol{T}(t) \\
 & \boldsymbol{T}: & \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^d), & \boldsymbol{x} \to \boldsymbol{T}(\boldsymbol{x}) \\
 & \boldsymbol{T}: & \mathbb{R}^d \times \mathbb{R} \to L(\mathbb{R}^d, \mathbb{R}^d), & (\boldsymbol{x}, t) \to \boldsymbol{T}(\boldsymbol{x}, t)
\end{array}$$

The notation $\mathbb{R}^d \times \mathbb{R}$ conveys that a field depends on two variables: one vector-valued and one scalar-valued variable. Examples for scalar fields include the temperature T, mass density ρ, entropy η, and energy density u. Vector fields characterize, among others, the velocity $\boldsymbol{v}$, acceleration $\boldsymbol{a}$, heat flux vector $\boldsymbol{h}$, and the traction vector $\boldsymbol{t}$. Tensor fields include, e.g., the deformation gradient $\boldsymbol{F}$ and the stress tensors $\boldsymbol{\sigma}$, $\boldsymbol{P}$ and $\boldsymbol{S}$.

As all these fields vary with position and time, let use define their derivatives. To this end, we introduce the *Gateaux derivative* $D\phi$ of a scalar field $\phi : \mathbb{R}^d \to \mathbb{R}$ at a point $\boldsymbol{x}$ into the direction of $\boldsymbol{v} \in \mathbb{R}^d$ (i.e., figuratively, the slope of ϕ at $\boldsymbol{x}$ into the direction of vector $\boldsymbol{v}$):

$$D\phi(\boldsymbol{x})\,\boldsymbol{v} = \left[\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\phi(\boldsymbol{x} + \varepsilon\,\boldsymbol{v})\right]_{\varepsilon=0}. \tag{99}$$

Let us apply this definition to the example $\tilde{\phi}(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{x}$, whose Gateaux derivative becomes

$$D\tilde{\phi}(\boldsymbol{x})\,\boldsymbol{v} = \left[\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\tilde{\phi}(\boldsymbol{x} + \varepsilon\,\boldsymbol{v})\right]_{\varepsilon=0} = \left[\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\boldsymbol{x} + \varepsilon\,\boldsymbol{v}) \cdot (\boldsymbol{x} + \varepsilon\,\boldsymbol{v})\right]_{\varepsilon=0} = 2\,\boldsymbol{v} \cdot (\boldsymbol{x} + \varepsilon\,\boldsymbol{v})\big|_{\varepsilon=0} = 2\,\boldsymbol{x} \cdot \boldsymbol{v}. \tag{100}$$

Now, we can use the concept of Gateaux derivatives to formally introduce the *gradient* of a scalar field $\phi(\boldsymbol{x})$ as a linear mapping $\operatorname{grad}\phi(\boldsymbol{x})$ which maps each direction $\boldsymbol{v}$ onto its Gateaux derivative $D\phi(\boldsymbol{x})\,\boldsymbol{v}$. Therefore, we define

$$\operatorname{grad}\phi(\boldsymbol{x}) \cdot \boldsymbol{v} = D\phi(\boldsymbol{x})\,\boldsymbol{v} = \left[\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\phi(\boldsymbol{x} + \varepsilon\,\boldsymbol{v})\right]_{\varepsilon=0}. \tag{101}$$

By using the chain rule and the summation convention, we see that

$$\operatorname{grad}\phi(\boldsymbol{x}) \cdot \boldsymbol{v} = \left[\frac{\partial\,\phi(\boldsymbol{x} + \varepsilon\,\boldsymbol{v})}{\partial\,(x_i + \varepsilon\,v_i)}\,\frac{\mathrm{d}(x_i + \varepsilon\,v_i)}{\mathrm{d}\varepsilon}\right]_{\varepsilon=0} = \frac{\partial\,\phi(\boldsymbol{x})}{\partial\,x_i}\,v_i, \tag{102}$$

but we also have (simply rewriting the inner product using index notation)

$$\operatorname{grad}\phi(\boldsymbol{x})\cdot\boldsymbol{v} = [\operatorname{grad}\phi(\boldsymbol{x})]_i\; v_i. \tag{103}$$

A direct comparison hence shows that the components of the gradient in Cartesian coordinates x_i $(i = 1,\ldots,d)$ are given by

$$[\operatorname{grad}\phi(\boldsymbol{x})]_i = \frac{\partial\phi(\boldsymbol{x})}{\partial x_i}. \tag{104}$$

Let us return to the above example $\tilde{\phi}(\boldsymbol{x}) = \boldsymbol{x}\cdot\boldsymbol{x}$ to clarify the use of the gradient. Here we have

$$\left[\operatorname{grad}\tilde{\phi}(\boldsymbol{x})\right]_i = \frac{\partial}{\partial x_i}(\boldsymbol{x}\cdot\boldsymbol{x}) = \frac{\partial}{\partial x_i}(x_k\,x_k) = 2\,x_k\,\delta_{ki} = 2\,x_i, \tag{105}$$

where we used that x_i $(i = 1,\ldots,d)$ are d independent coordinates so that

$$\frac{\partial x_k}{\partial x_i} = \delta_{ki}. \tag{106}$$

Altogether, for the example $\tilde{\phi}(\boldsymbol{x}) = \boldsymbol{x}\cdot\boldsymbol{x}$ we arrive at

$$\operatorname{grad}\tilde{\phi}(\boldsymbol{x}) = [\operatorname{grad}\phi(\boldsymbol{x})]_i\;\boldsymbol{g}_i = 2\,x_i\,\boldsymbol{g}_i = 2\,\boldsymbol{x}. \tag{107}$$

In continuum mechanics, one often uses the comma index notation to abbreviate partial derivatives; i.e., we define

$$\frac{\partial}{\partial x_i}(\cdot) = (\cdot)_{,i} \qquad \text{and likewise} \qquad \frac{\partial^2}{\partial x_i\,\partial x_j}(\cdot) = (\cdot)_{,ij} \quad \text{etc.} \tag{108}$$

so that, for example, (105) and (106) can be written in a concise way as

$$\tilde{\phi}_{,i} = 2\,x_i, \qquad x_{k,i} = \delta_{ki}. \tag{109}$$

The above definition of the gradient of a scalar field can be extended to general fields of higher order. Analogously, one can introduce the *divergence* and the *curl* of a field. Let us define all three for a *Cartesian* basis by

$$\operatorname{grad}(\cdot) = (\cdot)_{,i}\otimes\boldsymbol{g}_i, \qquad \operatorname{div}(\cdot) = (\cdot)_{,i}\cdot\boldsymbol{g}_i, \qquad \operatorname{curl}(\cdot) = -(\cdot)_{,i}\times\boldsymbol{g}_i. \tag{110}$$

where $(\cdot)$ stands for any tensor quantity of arbitrary order, and base vectors $\boldsymbol{g}_i$ act on the rightmost base vector of $(\cdot)$ via the respective operations. (The only exceptions arise for scalar fields, whose divergence and curl are not defined and whose gradient definition may omit the $\otimes$ symbol.) Let us clarify these definition by a few examples (all in a *Cartesian* reference frame):

- **gradient of a vector field**: let $\boldsymbol{v}:\mathbb{R}^d\to\mathbb{R}^d$ be differentiable, then we have

$$\operatorname{grad}\boldsymbol{v} = \frac{\partial\,v_i\,\boldsymbol{g}_i}{\partial\,x_j}\otimes\boldsymbol{g}_j = \frac{\partial\,v_i}{\partial\,x_j}\boldsymbol{g}_i\otimes\boldsymbol{g}_j \qquad\Rightarrow\qquad (\operatorname{grad}\boldsymbol{v})_{ij} = \frac{\partial\,v_i}{\partial\,x_j} = v_{i,j}. \tag{111}$$

  A classical example from solid mechanics is the definition of the infinitesimal strain tensor $\boldsymbol{\varepsilon} = \frac{1}{2}\left[\operatorname{grad}\boldsymbol{u} + (\operatorname{grad}\boldsymbol{u})^\mathsf{T}\right]$, which is a second-order tensor whose components are given by $\varepsilon_{ij} = \frac{1}{2}(u_{i,j}+u_{j,i})$.

- **divergence of a vector field**: let $\boldsymbol{v} : \mathbb{R}^d \to \mathbb{R}^d$ be differentiable, then we have

$$\operatorname{div} \boldsymbol{v} = \frac{\partial v_j \, \boldsymbol{g}_j}{\partial x_i} \cdot \boldsymbol{g}_i = \frac{\partial v_j}{\partial x_i} \boldsymbol{g}_j \cdot \boldsymbol{g}_i = v_{i,i}. \tag{112}$$

An alternative definition of the divergence of a vector field is given by

$$\operatorname{div} \boldsymbol{v} = \operatorname{tr} \left(\operatorname{grad} \boldsymbol{v}\right), \tag{113}$$

which can be verified to yield the identical expression as follows:

$$\operatorname{div} \boldsymbol{v} = \operatorname{tr} \left(\operatorname{grad} \boldsymbol{v}\right) = \operatorname{tr} \left(v_{i,j} \, \boldsymbol{g}_i \otimes \boldsymbol{g}_j\right) = v_{i,j} \, \boldsymbol{g}_j \cdot \boldsymbol{g}_i = v_{i,i}. \tag{114}$$

- **divergence of a tensor field**: let $\boldsymbol{T} : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ by differentiable, then we have

$$\operatorname{div} \boldsymbol{T} = \frac{\partial \, T_{jk} \, \boldsymbol{g}_j \otimes \boldsymbol{g}_k}{\partial x_i} \cdot \boldsymbol{g}_i = \frac{\partial \, T_{jk}}{\partial x_i} \, \boldsymbol{g}_j \left(\boldsymbol{g}_k \cdot \boldsymbol{g}_i\right) = \frac{\partial \, T_{jk}}{\partial x_i} \delta_{ik} \, \boldsymbol{g}_j = T_{ji,i} \, \boldsymbol{g}_j \tag{115}$$

and hence $(\operatorname{div} \boldsymbol{T})_i = T_{ij,j}$. An example from solid mechanics is the classical equilibrium equation, which reads in symbolic and indicial notation, respectively,

$$\operatorname{div} \boldsymbol{\sigma} + \rho \, \boldsymbol{b} = \rho \, \boldsymbol{a} \qquad \Rightarrow \qquad \sigma_{ij,j} + \rho \, b_i = \rho \, a_i, \tag{116}$$

where $\boldsymbol{\sigma}$ is the infinitesimal stress tensor, $\boldsymbol{b}$ is a vector of body forces, $\boldsymbol{a}$ denotes the acceleration vector, and ρ is the mass density.

- **curl of a vector field**: let $\boldsymbol{v} : \mathbb{R}^d \to \mathbb{R}^d$ be differentiable, then we have

$$\operatorname{curl} \boldsymbol{v} = -\frac{\partial \, v_i \, \boldsymbol{g}_i}{\partial x_j} \times \boldsymbol{g}_j = -\frac{\partial v_i}{\partial x_j} (\boldsymbol{g}_i \times \boldsymbol{g}_j) = -v_{i,j} \, \epsilon_{ijk} \, \boldsymbol{g}_k = v_{j,i} \, \epsilon_{ijk} \, \boldsymbol{g}_k \tag{117}$$

and so $(\operatorname{curl} \boldsymbol{v})_k = v_{j,i} \epsilon_{ijk}$.

- **curl of a tensor field**: let $\boldsymbol{T} : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ be differentiable, then we have

$$\operatorname{curl} \boldsymbol{T} = -\frac{\partial \, T_{kl} \, \boldsymbol{g}_k \otimes \boldsymbol{g}_l}{\partial x_j} \times \boldsymbol{g}_j = -\frac{\partial \, T_{kl}}{\partial x_j} \boldsymbol{g}_k \otimes (\boldsymbol{g}_l \times \boldsymbol{g}_j) = -T_{kl,j} \, \epsilon_{lji} \, \boldsymbol{g}_k \otimes \boldsymbol{g}_i = T_{kl,j} \, \epsilon_{jli} \, \boldsymbol{g}_k \otimes \boldsymbol{g}_i \tag{118}$$

and hence $(\operatorname{curl} \boldsymbol{T})_{ki} = T_{kl,j} \, \epsilon_{jli}$.

The gradient can furthermore be utilized to write the *total differential* of a scalar function $f : \mathbb{R} \to \mathbb{R}$ in Cartesian coordinates as

$$\mathrm{d}f = \frac{\partial f}{\partial x_i} \, \mathrm{d}x_i = \operatorname{grad} f \cdot \mathrm{d}\boldsymbol{x}. \tag{119}$$

We can also combine various of the above operators. For example, the *Laplacian* in Cartesian coordinates may be introduced as follows:

$$\Delta(\cdot) = \operatorname{div} \left[\operatorname{grad}(\cdot)\right] = (\cdot)_{,kk}. \tag{120}$$

Two helpful relations (that you will shown as part of the homework) are

$$\operatorname{curl}(\operatorname{grad} \phi) = \boldsymbol{0},$$
$$\operatorname{curl}(\operatorname{curl} \boldsymbol{v}) = \operatorname{grad}(\operatorname{div} \boldsymbol{v}) - \Delta \boldsymbol{v}.$$

Many authors make use the *del operator* $\boldsymbol{\nabla}$ which may be interpreted as a vector with components

$$\nabla_i = \partial/\partial x_i \tag{121}$$

in Cartesian coordinates. Then, it is common practice to introduce alternative abbreviations for the gradient, divergence and curl by using the del operator. Unfortunately, there is not a general convention of the definition and use of the del operator and most authors differ in their use of it. For example, defining $\boldsymbol{\nabla}$ as a vector with components $(\cdot)_{,i}$ turns (110) into $\operatorname{grad}(\cdot) = (\cdot) \otimes \boldsymbol{\nabla} = (\cdot)\boldsymbol{\nabla}$ as well as $\operatorname{div}(\cdot) = (\cdot) \cdot \boldsymbol{\nabla}$ and $\operatorname{curl}(\cdot) = \boldsymbol{\nabla} \times (\cdot)$. However, most continuum mechanicists define, e.g.,

$$\operatorname{grad}(\cdot) = \boldsymbol{\nabla}(\cdot), \qquad \operatorname{div}(\cdot) = \boldsymbol{\nabla} \cdot (\cdot). \tag{122}$$

For convenience, let us introduce a definition of the del operator which is consistent with (110) and (122) – note that this definition differs from the most common convention. For a consistent formulation, we may define the del operator (in an orthonormal basis) in the following way:

$$\boldsymbol{\nabla} \circ (\cdot) = \frac{\partial\,(\cdot)}{\partial\, x_j} \circ \boldsymbol{g}_j. \tag{123}$$

As before, $(\cdot)$ stands for any arbitrary (differentiable) tensor quantity (scalars, vectors, higher-order tensors). Here, $\circ$ represents any tensor product operation defined above (such as, e.g., inner product, outer product or cross product). Then, we conclude by comparison with our original definition of gradient, divergence and curl that we must define

$$\operatorname{grad}(\cdot) = \boldsymbol{\nabla} \otimes (\cdot), \qquad \operatorname{div}(\cdot) = \boldsymbol{\nabla} \cdot (\cdot), \qquad \operatorname{curl}(\cdot) = -\boldsymbol{\nabla} \times (\cdot) \tag{124}$$

Clearly the simplest and unambiguous formulation avoids of the del operator overall and instead only uses the terms $\operatorname{grad}\boldsymbol{v}$, $\operatorname{div}\boldsymbol{v}$ and $\operatorname{curl}\boldsymbol{v}$, which is what we will do in the following.

Table 2 completes the summary of Table 1 by the notions of gradient, divergence and curl of tensors of scalar, vector and tensor fields.

1.2 General Tensor Derivatives

So far, all derivatives have been with respect to scalar or vector-valued variables. Let us extend the concepts introduced above to derivatives with respect to general tensors of arbitrary order.

symbolic notation	full component form	indicial notation
$\operatorname{grad}\phi$	$\phi_{,i}\boldsymbol{g}_i$	$[\operatorname{grad}\phi]_i = \phi_{,i}$
$\operatorname{grad}\boldsymbol{v}$	$v_{i,j}\boldsymbol{g}_i \otimes \boldsymbol{g}_j$	$[\operatorname{grad}\boldsymbol{v}]_{ij} = v_{i,j}$
$\operatorname{grad}\boldsymbol{T}$	$T_{ij,k}\boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}_k$	$[\operatorname{grad}\boldsymbol{T}]_{ijk} = T_{ij,k}$
$\operatorname{div}\boldsymbol{v}$	$v_{i,i}$	$v_{i,i}$
$\operatorname{div}\boldsymbol{T}$	$T_{ij,j}\boldsymbol{g}_i$	$[\operatorname{div}\boldsymbol{T}]_i = T_{ij,j}$
$\operatorname{curl}\boldsymbol{v}$	$v_{i,j}\epsilon_{jik}\boldsymbol{g}_k$	$[\operatorname{curl}\boldsymbol{v}]_k = v_{i,j}\epsilon_{jik}$
$\operatorname{curl}\boldsymbol{T}$	$T_{ik,l}\epsilon_{lkj}\boldsymbol{g}_i \otimes \boldsymbol{g}_j$	$[\operatorname{curl}\boldsymbol{T}]_{ij} = T_{ik,l}\epsilon_{lkj}$
$\operatorname{div}(\operatorname{grad}\boldsymbol{v})$	$v_{i,kk}\boldsymbol{g}_i$	$[\operatorname{div}(\operatorname{grad}\boldsymbol{v})]_i = v_{i,kk}$

Table 2: Overview of gradient, divergence and curl of scalars, vectors and tensors in symbolic and indicial notation where we assume differentiable fields $\phi : \mathbb{R}^d \to \mathbb{R}$, $\boldsymbol{v} : \mathbb{R}^d \to \mathbb{R}^d$, and $\boldsymbol{T} : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$.

First, we introduce the alternative notation

$$\frac{\partial(\cdot)}{\partial \boldsymbol{x}} = \text{grad}(\cdot) = \frac{\partial(\cdot)}{\partial x_j} \otimes \boldsymbol{g}_j. \tag{125}$$

In an analogous fashion, let us introduce the derivative with respect to a general vector $\boldsymbol{v} \in \mathbb{R}^d$ as

$$\frac{\partial(\cdot)}{\partial \boldsymbol{v}} = \frac{\partial(\cdot)}{\partial v_j} \otimes \boldsymbol{g}_j, \tag{126}$$

where, again, $(\cdot)$ stands for any tensor quantity of arbitrary order. In an extension of (106), we concluded that for tensors of arbitrary order we have

$$\frac{\partial T_{ij\dots n}}{\partial T_{ab\dots m}} = \delta_{ia}\,\delta_{jb}\,\dots\,\delta_{nm}. \tag{127}$$

As a quick example, let us assume that $\boldsymbol{T} \in L(\mathbb{R}^d, \mathbb{R}^d)$ is constant and $\boldsymbol{v} \in \mathbb{R}^d$ so that

$$\frac{\partial \boldsymbol{T}\boldsymbol{v}}{\partial \boldsymbol{v}} = \frac{\partial\, T_{ij} v_j\, \boldsymbol{g}_i}{\partial\, v_k} \otimes \boldsymbol{g}_k = T_{ij}\frac{\partial\, v_j}{\partial\, v_k}\boldsymbol{g}_i \otimes \boldsymbol{g}_k = T_{ij}\,\delta_{jk}\,\boldsymbol{g}_i \otimes \boldsymbol{g}_k = T_{ij}\,\boldsymbol{g}_i \otimes \boldsymbol{g}_j = \boldsymbol{T}. \tag{128}$$

Definition (126) can be generalized to tensors of arbitrary order by defining

$$\frac{\partial\,(\cdot)}{\partial\,\boldsymbol{S}} = \frac{\partial\,(\cdot)}{\partial\,S_{ij\dots n}} \otimes \boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \ldots \otimes \boldsymbol{g}_n \tag{129}$$

(where, as before for the gradient, the leading $\otimes$ must be omitted when applied to scalar fields). As an example, let us take the derivative of the trace of a tensor $\boldsymbol{T} \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ with respect to that tensor, i.e.,

$$\frac{\partial\ \text{tr}\ \boldsymbol{T}}{\partial\,\boldsymbol{T}} = \frac{\partial\,T_{kk}}{\partial\,T_{ij}}\boldsymbol{g}_i \otimes \boldsymbol{g}_j = \delta_{ik}\,\delta_{jk}\,\boldsymbol{g}_i \otimes \boldsymbol{g}_j = \delta_{ij}\,\boldsymbol{g}_i \otimes \boldsymbol{g}_j = \boldsymbol{I}. \tag{130}$$

1.3 Product Rule and Chain Rule

By applying the aforementioned rules of tensor differentiation, we notice that, when taking derivatives of products of tensor quantities, the *product rule* of basic calculus can be generalized here. In general, we may write

$$\frac{\partial\,\boldsymbol{\Phi} \circ \boldsymbol{\Psi}}{\partial\,\boldsymbol{T}} = \frac{\partial\,\boldsymbol{\Phi}}{\partial\,\boldsymbol{T}} \circ \boldsymbol{\Psi} + \boldsymbol{\Phi} \circ \frac{\partial\,\boldsymbol{\Psi}}{\partial\,\boldsymbol{T}}, \tag{131}$$

where $\boldsymbol{\Phi}$, $\boldsymbol{\Psi}$ and $\boldsymbol{T}$ are tensor quantities of arbitrary order, and $\circ$ is any tensor multiplication (inner, outer, or cross product). Let us illustrate the use of the product rule by the following example:

$$\text{div}(\boldsymbol{T}\boldsymbol{v}) = (T_{ij}\,v_j)_{,i} = T_{ij,i}\,v_j + T_{ij}\,v_{j,i}, \tag{132}$$

which may be written in symbolic notation as (recall the definition $\boldsymbol{S} \cdot \boldsymbol{T} = S_{ij}\,T_{ij}$)

$$T_{ij,i}\,v_j + T_{ij}\,v_{j,i} = (\text{div}\,\boldsymbol{T}^{\mathsf{T}}) \cdot \boldsymbol{v} + \boldsymbol{T} \cdot (\text{grad}\,\boldsymbol{v})^{\mathsf{T}}. \tag{133}$$

As a further example, let us show that the curl of a gradient vector field vanishes. For $\phi : \mathbb{R}^d \to \mathbb{R}$ we see that

$$\operatorname{curl}(\operatorname{grad}\phi) = -(\phi_{,i}\,\boldsymbol{g}_i)_{,j} \times \boldsymbol{g}_j = -\phi_{,ij}\,\boldsymbol{g}_i \times \boldsymbol{g}_j = \phi_{,ij}\,\epsilon_{jik}\,\boldsymbol{g}_k. \tag{134}$$

Knowing that we are allowed to exchange the order of derivatives in $\phi_{,ij}$ and renaming indices i and j, we arrive at

$$\phi_{,ij}\,\epsilon_{jik}\,\boldsymbol{g}_k = \phi_{,ji}\,\epsilon_{jik}\,\boldsymbol{g}_k = \phi_{,ij}\,\epsilon_{ijk}\,\boldsymbol{g}_k = -\phi_{,ij}\,\epsilon_{jik}\,\boldsymbol{g}_k. \tag{135}$$

Comparing the first and final forms, we conclude that $\phi_{,ij} = -\phi_{,ij} = 0$ and therefore

$$\operatorname{curl}(\operatorname{grad}\phi) = \mathbf{0}. \tag{136}$$

The *chain rule* of differentiation, known from fundamental calculus, also holds for general tensor differentiation. As an introductory example, let us consider a scalar field $\phi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ which depends on the vector field $\boldsymbol{x} : \mathbb{R} \to \mathbb{R}^d$ and scalar $t \in \mathbb{R}$ (e.g., $\phi = \phi(\boldsymbol{x}, t)$ denotes the temperature at Eulerian position $\boldsymbol{x}(t)$ at time t). That is, we consider the composition

$$\phi\left(\boldsymbol{x}(t), t\right) = \phi\left(x_1(t), x_2(t), \ldots, x_n(t), t\right). \tag{137}$$

The total differential is then given by

$$\mathrm{d}\phi = \frac{\partial \phi}{\partial t}\,\mathrm{d}t + \frac{\partial \phi}{\partial x_i}\,\mathrm{d}x_i = \frac{\partial \phi}{\partial t}\,\mathrm{d}t + \frac{\partial \phi}{\partial \boldsymbol{x}} \cdot \mathrm{d}\boldsymbol{x}. \tag{138}$$

Next, let us consider the following example which we first solve directly without using the chain rule but by utilizing the product rule:

$$\begin{aligned}\frac{\partial\,\operatorname{tr}\left(\boldsymbol{A}^\mathsf{T}\boldsymbol{A}\right)}{\partial\,\boldsymbol{A}} &= \frac{\partial\,A_{ij}\,A_{ij}}{\partial\,A_{kl}}\,\boldsymbol{g}_k \otimes \boldsymbol{g}_l = \left(\delta_{ik}\,\delta_{jl}\,A_{ij} + A_{ij}\,\delta_{ik}\,\delta_{jl}\right)\,\boldsymbol{g}_k \otimes \boldsymbol{g}_l \\ &= 2\,A_{ij}\,\boldsymbol{g}_i \otimes \boldsymbol{g}_j = 2\,\boldsymbol{A}.\end{aligned} \tag{139}$$

Now let us revisit the same example and find the solution by using the chain rule. To this end, let us define $\boldsymbol{B} = \boldsymbol{A}^\mathsf{T}\boldsymbol{A}$ or $B_{ij} = A_{ki}A_{kj}$. Then we have

$$\begin{aligned}\frac{\partial\,\operatorname{tr}\left(\boldsymbol{A}^\mathsf{T}\boldsymbol{A}\right)}{\partial\boldsymbol{A}} &= \frac{\partial\,\operatorname{tr}\,\boldsymbol{B}}{\partial\boldsymbol{A}} = \frac{\partial B_{nn}}{\partial B_{ij}}\,\frac{\partial B_{ij}}{\partial A_{kl}}\,\boldsymbol{g}_k \otimes \boldsymbol{g}_l = \delta_{ij}\,\frac{\partial\,A_{ni}A_{nj}}{\partial\,A_{kl}}\boldsymbol{g}_k \otimes \boldsymbol{g}_l \\ &= \delta_{ij}\,\left(\delta_{nk}\delta_{il}A_{nj} + A_{ni}\delta_{nk}\delta_{jl}\right)\boldsymbol{g}_k \otimes \boldsymbol{g}_l = 2\,A_{kl}\,\boldsymbol{g}_k \otimes \boldsymbol{g}_l = 2\,\boldsymbol{A}.\end{aligned}$$

As a final example, often required in plasticity theory, consider a scalar field $W : L(\mathbb{R}^d, \mathbb{R}^d) \to \mathbb{R}$ and tensors $\boldsymbol{F}, \boldsymbol{Q} \in L(\mathbb{R}^d, \mathbb{R}^d)$. The chain rule yields the useful relation

$$\frac{\partial W(\boldsymbol{FQ})}{\partial \boldsymbol{F}} = \frac{\partial W(\boldsymbol{FQ})}{\partial \boldsymbol{FQ}} \cdot \frac{\partial \boldsymbol{FQ}}{\partial \boldsymbol{F}} = \frac{\partial W(\boldsymbol{FQ})}{\partial \boldsymbol{FQ}}\boldsymbol{Q}^\mathsf{T}. \tag{140}$$

1.4 Divergence Theorem and Kelvin–Stokes Theorem

In continuum mechanics, we often integrate scalar, vector and tensor fields over volumes (e.g. over the volume of a body of solid or fluid). It is convenient to introduce two important theorems that allow us to transform volume integrals into surface integrals and vice-versa.

First, the *divergence theorem* (also referred to as *Gauß' theorem* after German mathematician Johann Carl Friedrich Gauß) reads

$$\int_V (\cdot)_{,j}\,\mathrm{d}v = \int_{\partial V} (\cdot)\,n_j\,\mathrm{d}s, \qquad \text{e.g.} \qquad \int_V T_{ij,j}\,\mathrm{d}v = \int_{\partial V} T_{ij}\,n_j\,\mathrm{d}s \tag{141}$$

As before $(\cdot)$ stands for a tensor quantity of arbitrary order, and $\boldsymbol{n} = n_i\,\boldsymbol{g}_i$ denotes the outward unit normal on the surface ∂V of body V. Let us consider the example of the divergence of a tensor (which we will need later on when discussing the conditions of equilibrium):

$$\int_V \operatorname{div}\boldsymbol{\sigma}\,\mathrm{d}v = \int_V \sigma_{ij,j}\,\mathrm{d}v = \int_{\partial V} \sigma_{ij}\,n_j\,\mathrm{d}s = \int_{\partial V} \boldsymbol{\sigma}\boldsymbol{n}\,\mathrm{d}s, \tag{142}$$

where $\boldsymbol{\sigma} \in L(\mathbb{R}^d, \mathbb{R}^d)$ is the infinitesimal stress tensor.

In a similar fashion, the *Kelvin–Stokes theorem* (named after English scientist Lord Kelvin and Irish contemporary Sir George Stokes) relates surface and contour integrals. The integral of the curl of a second-order tensor $\boldsymbol{T} \in L(\mathbb{R}^d, \mathbb{R}^d)$ over a surface S is transformed into an integral over the contour ∂A of that surface:

$$\int_S (\operatorname{curl}\boldsymbol{T})\boldsymbol{n}\,\mathrm{d}s = \int_{\partial S} \boldsymbol{T}\,\mathrm{d}\boldsymbol{x} \qquad \text{or} \qquad \int_S T_{ij,k}\,\epsilon_{kjl}\,n_l\,\mathrm{d}s = \int_{\partial S} T_{ij}\,\mathrm{d}x_j. \tag{143}$$

2 Curvilinear Coordinates

2.1 Covariant and Contravariant Basis Representations

So far we have only dealt with Cartesian coordinate systems in which the base vectors are orthonormal and constant in space and time. This may not always be convenient. For example, spherical or circular bodies suggest using spherical or cylindrical coordinate systems, respectively, in which base vectors change orientation from one point to another. As a further example, one might want to align coordinate axes with specific directions of a mechanical problem, which may in general not be perpendicular. In all such cases, the above tensor relations no longer apply and we must introduce a generalized framework for general curvilinear bases. To understand the necessity of new tensor rules, let us begin by reviewing the relations that hold for an *orthonormal basis*. We had seen that the components of a vector $\boldsymbol{v} \in \mathbb{R}^d$ in a Cartesian basis can be obtained by exploiting the relation $\boldsymbol{g}_i \cdot \boldsymbol{g}_j = \delta_{ij}$:

$$\boldsymbol{v} = v_i\,\boldsymbol{g}_i \qquad \Rightarrow \qquad \boldsymbol{v}\cdot\boldsymbol{g}_j = v_i\,\delta_{ij} = v_j \qquad \Leftrightarrow \qquad v_i = \boldsymbol{v}\cdot\boldsymbol{g}_i. \tag{144}$$

This further allows us to identify the identity tensor in an orthonormal basis as follows:

$$\boldsymbol{v} = v_i\,\boldsymbol{g}_i = (\boldsymbol{v}\cdot\boldsymbol{g}_i)\,\boldsymbol{g}_i = \boldsymbol{v}\cdot(\boldsymbol{g}_i \otimes \boldsymbol{g}_i) \qquad \Leftrightarrow \qquad \boldsymbol{I} = \boldsymbol{g}_i \otimes \boldsymbol{g}_i \tag{145}$$

All of this holds because of the important relation $\boldsymbol{g}_i \cdot \boldsymbol{g}_j = \delta_{ij}$.

Let us now turn to a general *curvilinear coordinate system*, in which the base vectors are no longer perpendicular and may not have unit lengths either, i.e., from now on we have

$$\boldsymbol{g}_i \cdot \boldsymbol{g}_j \neq \delta_{ij}, \qquad |\boldsymbol{g}_i| \neq 1. \tag{146}$$

To avoid confusion, in the following we will denote the base vectors in general curvilinear coordinates by $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$, so that in our notation the symbol $\boldsymbol{g}$ is reserved for orthonormal bases. Now, the above concepts and, in particular, relation (144) fails because we no longer have $\boldsymbol{a}_i \cdot \boldsymbol{a}_j = \delta_{ij}$, which was the basis of determining vector components. Instead, $\boldsymbol{v} \cdot \boldsymbol{a}_j = v_i\, \boldsymbol{a}_i \cdot \boldsymbol{a}_j$ and no further simplification seems possible, which would force us to review and correct all of the vector and tensor operations introduced before for their use in a general curvilinear basis. As a remedy, let us introduce a dual basis in the following way. For a given curvilinear basis $\mathcal{B} = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ we construct a *dual basis* $\mathcal{B}' = \left\{\boldsymbol{a}^1, \boldsymbol{a}^2, \boldsymbol{a}^3\right\}$ by defining

$$\boldsymbol{a}^1 = \frac{\boldsymbol{a}_2 \times \boldsymbol{a}_3}{\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3)}, \qquad \boldsymbol{a}^2 = \frac{\boldsymbol{a}_3 \times \boldsymbol{a}_1}{\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3)}, \qquad \boldsymbol{a}^3 = \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3)}. \tag{147}$$

Notice that these definitions yield the following appealing properties:

- $\boldsymbol{a}_1 \cdot \boldsymbol{a}^1 = \boldsymbol{a}_2 \cdot \boldsymbol{a}^2 = \boldsymbol{a}_3 \cdot \boldsymbol{a}^3 = 1$,
- $\boldsymbol{a}_1 \cdot \boldsymbol{a}^2 = \boldsymbol{a}_2 \cdot \boldsymbol{a}^1 = \boldsymbol{a}_2 \cdot \boldsymbol{a}^3 = \boldsymbol{a}_3 \cdot \boldsymbol{a}^2 = \boldsymbol{a}_1 \cdot \boldsymbol{a}^3 = \boldsymbol{a}_3 \cdot \boldsymbol{a}^1 = 0$.

These two facts can neatly be summarized by defining a modified Kronecker delta

$$\boldsymbol{a}_i \cdot \boldsymbol{a}^j = \delta_i^j, \qquad \boldsymbol{a}_j \cdot \boldsymbol{a}^i = \delta_j^i \tag{148}$$

with

$$\delta_i^j = \boldsymbol{a}_i \cdot \boldsymbol{a}^j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{149}$$

In the following, let us show that our newly-constructed dual basis is a *reciprocal basis* to our original base vectors, i.e., if we apply the very same definitions (147) (that we introduced to arrive at the dual basis) to the dual basis $\mathcal{B}'$ to obtain its dual $\mathcal{B}''$, we should return to the original basis $\mathcal{B}$. First, verify that

$$\begin{aligned} \boldsymbol{a}^1 \cdot (\boldsymbol{a}^2 \times \boldsymbol{a}^3) &= \frac{\boldsymbol{a}_2 \times \boldsymbol{a}_3}{\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3)} \cdot \left[\frac{\boldsymbol{a}_3 \times \boldsymbol{a}_1}{\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3)} \times \frac{\boldsymbol{a}_1 \times \boldsymbol{a}_2}{\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3)}\right] \\ &= (\boldsymbol{a}_2 \times \boldsymbol{a}_3) \cdot \frac{(\boldsymbol{a}_1 \times \boldsymbol{a}_2) \times (\boldsymbol{a}_1 \times \boldsymbol{a}_3)}{[\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3)]^3} = (\boldsymbol{a}_2 \times \boldsymbol{a}_3) \cdot \frac{[\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3)]\, \boldsymbol{a}_1}{[\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3)]^3} \\ &= \frac{1}{\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3)}, \end{aligned}$$

which is a requirement for a reciprocal basis (the triple product of our dual basis is the inverse of the triple product of the original basis). Hence, if we construct a dual basis to our dual basis in the same way as above, we arrive at a basis whose triple product is exactly the same as the one of our original basis. This becomes clearer when constructing a dual basis $\mathcal{B}''$ to our dual basis $\mathcal{B}$. Let us, for example, take vector $\boldsymbol{a}^3$ of our dual basis and use the above relations to introduce a reciprocal base vector in the following way. We had seen that the dual basis $\mathcal{B}'$ to $\mathcal{B}$ was obtained via (147). Let us compute the dual basis $\mathcal{B}''$ to basis $\mathcal{B}'$ via the same relations, e.g. applying the last relation of (147) to $\mathcal{B}' = \left\{\boldsymbol{a}^1, \boldsymbol{a}^2, \boldsymbol{a}^3\right\}$:

$$\frac{\boldsymbol{a}^1 \times \boldsymbol{a}^2}{\boldsymbol{a}^1 \cdot (\boldsymbol{a}^2 \times \boldsymbol{a}^3)} = [\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3)]\, \frac{(\boldsymbol{a}_2 \times \boldsymbol{a}_3) \times (\boldsymbol{a}_3 \times \boldsymbol{a}_1)}{[\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3)]^2} = \frac{[\boldsymbol{a}_3 \cdot \times (\boldsymbol{a}_1 \times \boldsymbol{a}_2)]\, \boldsymbol{a}_3}{\boldsymbol{a}_1 \cdot (\boldsymbol{a}_2 \times \boldsymbol{a}_3)} = \boldsymbol{a}_3. \tag{150}$$

Hence, we see that the dual basis to our dual basis is in fact the original basis (the analogous can be shown for the other two base vectors). This means that we have constructed two reciprocal bases $\mathcal{B}$ and $\mathcal{B}'$ which can be translated from one into another via relations (147).

Let us return to the Kronecker property which no longer holds for base vectors of either of the two bases, i.e., we have $\boldsymbol{a}_i \cdot \boldsymbol{a}_j \neq \delta_{ij}$ and likewise $\boldsymbol{a}^i \cdot \boldsymbol{a}^j \neq \delta_{ij}$. As we will need those inner products frequently, it is convenient to define

$$\boldsymbol{a}_i \cdot \boldsymbol{a}_j = g_{ij}, \qquad \boldsymbol{a}^i \cdot \boldsymbol{a}^j = g^{ij}, \tag{151}$$

where g_{ij} and g^{ij} are the components of the so-called *metric tensors* of the original and dual bases, respectively. The determinant of the metric tensor is denoted by

$$g = \det[g_{ij}]. \tag{152}$$

Two special cases can be identified in which the above concept simplifies greatly. First, let us return to the simplest case of a Cartesian coordinate system (that we had dealt with before). When using a Cartesian basis, our new concept of reciprocal bases should recover all the relations we had introduced in previous sections. Indeed, for a *Cartesian basis* $\mathcal{B} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3\}$ we can use relations (147) to show that

$$\{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3\} = \left\{\boldsymbol{g}^1, \boldsymbol{g}^2, \boldsymbol{g}^3\right\} \qquad \text{and} \qquad g_{ij} = g^{ij} = \delta_{ij}. \tag{153}$$

The first fact can easily be verified by constructing the dual basis, e.g.,

$$\boldsymbol{g}^1 = \frac{\boldsymbol{g}_2 \times \boldsymbol{g}_3}{\boldsymbol{g}_1 \cdot (\boldsymbol{g}_2 \times \boldsymbol{g}_3)} = \frac{\boldsymbol{g}_1}{1} = \boldsymbol{g}_1 \tag{154}$$

and similarly $\boldsymbol{g}^2 = \boldsymbol{g}_2$ and $\boldsymbol{g}^3 = \boldsymbol{g}_3$. The second relation follows directly because $g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j = \delta_{ij}$ in any orthonormal basis. In conclusion, an orthonormal basis is a special case in which the concept of dual bases reduces to our well-known formulation from previous sections.

The second special case we consider here comprises *orthogonal bases* whose base vectors are not necessarily of unit length (this will become important when dealing with, e.g., polar coordinate systems). In this case, base vectors are still mutually perpendicular (i.e., $\boldsymbol{a}_i \cdot \boldsymbol{a}_j = 0$ if $i \neq j$, and $\boldsymbol{a}^i \cdot \boldsymbol{a}^j = 0$ if $i \neq j$) but we no longer have $\boldsymbol{a}_i \cdot \boldsymbol{a}_i = 1$ or $\boldsymbol{a}^i \cdot \boldsymbol{a}^i = 1$. Consequently, the metric tensor $[g_{ij}]$ is diagonal but not the identity. It will be convenient to introduce a new symbol for those diagonal components; to this end we define the *scale factors*

$$h_{(i)} = \sqrt{\boldsymbol{a}_{(i)} \cdot \boldsymbol{a}_{(i)}}. \tag{155}$$

As before, the parentheses are included to suppress summation (i.e., the above relation holds for all $i = 1, 2, 3$). In summary, the metric tensor for an orthogonal basis assumes the form

$$[g_{ij}] = \begin{bmatrix} \boldsymbol{a}_1 \cdot \boldsymbol{a}_1 & 0 & 0 \\ 0 & \boldsymbol{a}_2 \cdot \boldsymbol{a}_2 & 0 \\ 0 & 0 & \boldsymbol{a}_3 \cdot \boldsymbol{a}_3 \end{bmatrix} = \begin{bmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{bmatrix}. \tag{156}$$

For the special case of an orthonormal basis we have $h_1 = h_2 = h_3 = 1$ and consequently $g_{ij} = \delta_{ij}$.

Now, we have all tools required to solve our original problem of computing the components of a vector in curvilinear coordinates. As our two reciprocal bases are mutually orthogonal (recall that $\boldsymbol{a}_i \cdot \boldsymbol{a}^j = \delta_i^j$), we can obtain the component of vector $\boldsymbol{v}$ in the $\boldsymbol{a}_j$-direction from $\boldsymbol{v} \cdot \boldsymbol{a}^j$, and we can obtain the component of vector $\boldsymbol{v}$ in the $\boldsymbol{a}^j$-direction from $\boldsymbol{v} \cdot \boldsymbol{a}_j$. As a consequence, we can write

$$\boldsymbol{v} = (\boldsymbol{v} \cdot \boldsymbol{a}^1)\, \boldsymbol{a}_1 + (\boldsymbol{v} \cdot \boldsymbol{a}^2)\, \boldsymbol{a}_2 + (\boldsymbol{v} \cdot \boldsymbol{a}^3)\, \boldsymbol{a}_3 = (\boldsymbol{v} \cdot \boldsymbol{a}^i)\, \boldsymbol{a}_i \tag{157}$$

and analogously

$$\boldsymbol{v} = (\boldsymbol{v} \cdot \boldsymbol{a}_1)\, \boldsymbol{a}^1 + (\boldsymbol{v} \cdot \boldsymbol{a}_2)\, \boldsymbol{a}^2 + (\boldsymbol{v} \cdot \boldsymbol{a}_3)\, \boldsymbol{a}^3 = (\boldsymbol{v} \cdot \boldsymbol{a}_i)\, \boldsymbol{a}^i. \tag{158}$$

The last two forms reflect exactly the definition of a vector in its base representation as we know it (and for a Cartesian base, where the reciprocal basis coincides with the original basis, we recover equation (2)). Note that the summation convention over repeated subscripts/superscripts applies as before.

To differentiate between the two different types of base vectors and components defined by these relations, we introduce the following terms:

$$\begin{aligned}
\textit{contravariant components}: &\quad v^i = \boldsymbol{v} \cdot \boldsymbol{a}^i \\
\textit{covariant components}: &\quad v_i = \boldsymbol{v} \cdot \boldsymbol{a}_i \\
\textit{contravariant basis}: &\quad \left\{\boldsymbol{a}^1, \boldsymbol{a}^2, \boldsymbol{a}^3\right\} \\
\textit{covariant basis}: &\quad \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}
\end{aligned}$$

Hence, a vector can be written in its base representation by using contravariant components with covariant base vectors, or by using covariant components and contravariant base vectors:

$$\boldsymbol{v} = v^i\, \boldsymbol{a}_i = v_i\, \boldsymbol{a}^i. \tag{159}$$

Besides, from equations (157) and (158) we recover the completeness relation as

$$\boldsymbol{I} = \boldsymbol{a}_i \otimes \boldsymbol{a}^i \qquad \text{or} \qquad \boldsymbol{I} = \boldsymbol{a}^i \otimes \boldsymbol{a}_i. \tag{160}$$

We note the relatedness to notation commonly used in quantum mechanics where bras and kets act equivalently to our covariant and contravariant forms.

Finally, let us complete this section by reviewing the inner product of two vectors. With our new definitions we can write the inner product in four different ways:

$$\begin{aligned}
\boldsymbol{u} \cdot \boldsymbol{v} &= u^i\, \boldsymbol{a}_i \cdot v^j\, \boldsymbol{a}_j = u^i\, v^j\, g_{ij} \\
&= u_i\, \boldsymbol{a}^i \cdot v_j\, \boldsymbol{a}^j = u_i\, v_j\, g^{ij} \\
&= u^i\, \boldsymbol{a}_i \cdot v_j\, \boldsymbol{a}^j = u^i\, v_j\, \delta_i^j = u^i\, v_i \\
&= u_i\, \boldsymbol{a}^i \cdot v^j\, \boldsymbol{a}_j = u_i\, v^j\, \delta_j^i = u_i\, v^i.
\end{aligned}$$

In the first two forms it is important to not forget the components of the metric tensor to obtain the correct result. Consequently, the norm in a curvilinear basis is given by $|\boldsymbol{u}| = \sqrt{u_i\, u_j\, g^{ij}}$, which in a Cartesian basis with $g^{ij} = \delta_{ij}$ reduces to (13).

2.2 Derivatives in general orthogonal coordinates, physical components and basis

Many problems in continuum mechanics are conveniently solved, e.g., in cylindrical or spherical polar coordinates. In these coordinate systems our above definitions of, e.g., the gradient, divergence and curl no longer apply because base vectors depend on the position and are no longer constant. Furthermore, the nature of coordiantes may no longer be compatible. For example, in Cartesian coordinates (x_1, x_2, x_3) we introduced the gradient of a scalar field ϕ as $\operatorname{grad}\phi = (\phi_{,1}, \phi_{,2}, \phi_{,3})^\mathsf{T}$. In contrast, in cylindrical polar coordinates (r, φ, z) we may not write $\operatorname{grad}\phi$ in as $(\phi_{,r}, \phi_{,\varphi}, \phi_{,z})^\mathsf{T}$. This does not make much sense (consider e.g. the non-matching units). In this section, we will extend our previous concepts of differentiation to more general coordinate bases. For simplicity, we will restrict ourselves to *orthogonal bases* only (which includes, e.g., polar coordinates of any type as well as elliptical coordinates).

In any orthogonal coordinate system, we characterize points in space by d coordinates which we term θ^i (we reserve the symbol x^i for Cartesian reference frames only). Vector $\boldsymbol{x}$ in its base representation using the Cartesian basis $\{\boldsymbol{e}_i\}$ is then written as, respectively,

$$\boldsymbol{x} = \sum_i x^i\, \boldsymbol{e}_i \qquad \text{so that} \qquad \boldsymbol{e}_i = \frac{\partial \boldsymbol{x}}{\partial x_i}. \tag{161}$$

Analogously, we can define a general (covariant) basis $\{\boldsymbol{a}_i\}$ (recalling that the Cartesian basis is independent of θ_i) by

$$\boldsymbol{a}_i = \frac{\partial \boldsymbol{x}}{\partial \theta_i} = \frac{\partial}{\partial \theta^i} \sum_{j=1}^{d} x^j\, \boldsymbol{e}_j = \sum_{j=1}^{d} \frac{\partial x^j}{\partial \theta^i}\, \boldsymbol{e}_j. \tag{162}$$

Let us illustrate this by an example: cylindrical polar coordinates (r, φ, z). We know the relation between cylindrical polar coordinates and Cartesian coordinates, i.e., the location of a point characterized by radius r, angle φ and out-of-plane coordinate z with respect to some origin O is given in Cartesian coordinates (where there is no difference between covariant and contravariant bases) as

$$x_1 = r\,\cos\varphi, \qquad x_2 = r\,\sin\varphi, \qquad x_3 = z. \tag{163}$$

So, we can derive the *covariant base vectors* as

$$\begin{aligned}
\boldsymbol{a}_r &= \frac{\partial}{\partial r}\sum_{j=1}^{d} x_j\, \boldsymbol{e}_j \qquad &\Rightarrow \qquad [\boldsymbol{a}_r] &= (\cos\varphi, \sin\varphi, 0)^\mathsf{T},\\
\boldsymbol{a}_\varphi &= \frac{\partial}{\partial \varphi}\sum_{j=1}^{d} x_j\, \boldsymbol{e}_j \qquad &\Rightarrow \qquad [\boldsymbol{a}_\varphi] &= (-r\,\sin\varphi, r\,\cos\varphi, 0)^\mathsf{T},\\
\boldsymbol{a}_z &= \frac{\partial}{\partial z}\sum_{j=1}^{d} x_j\, \boldsymbol{e}_j \qquad &\Rightarrow \qquad [\boldsymbol{a}_z] &= (0, 0, 1)^\mathsf{T},
\end{aligned} \tag{164}$$

It is important to note that these base vectors are no longer constant throughout space (they clearly depend on coordinates r, φ and z), i.e., we now have a dependence of the type $\boldsymbol{a}_i = \boldsymbol{a}_i(\theta^1, \theta^2, \theta^3)$ that will be essential soon.

Unfortunately, the thus-obtained covairnat basis has one major deficiency which makes it inconvenient for practical applications: it is not normalized. As a check, compute the norm of all three base vectors or simply compute the components of the metric tensor corresponding to the above basis:

$$[a_{ij}] = [\boldsymbol{a}_i \cdot \boldsymbol{a}_j] = \begin{bmatrix} \boldsymbol{a}_r \cdot \boldsymbol{a}_r & 0 & 0 \\ 0 & \boldsymbol{a}_\varphi \cdot \boldsymbol{a}_\varphi & 0 \\ 0 & 0 & \boldsymbol{a}_z \cdot \boldsymbol{a}_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{165}$$

Hence, $\boldsymbol{a}_r$ and $\boldsymbol{a}_z$ have unit length, whereas $\boldsymbol{a}_\varphi$ has length r. Recall that those diagonal entries of the metric tensor had been introduced above as the (squared) scale factors h_i^2. In order to create an orthonormal system, let us introduce an orthonormal basis that will correspond to what we call the *physical components* of a vector. This is accomplished by dividing each base vector by its corresponding scale factor, i.e., we introduce the orthonormal *physical basis* $\{\boldsymbol{g}_i\}$ as

$$\boldsymbol{g}_i = \frac{\boldsymbol{a}_{(i)}}{h_{(i)}}. \tag{166}$$

This means for our example of polar coordinates

$$\{\boldsymbol{g}_r, \boldsymbol{g}_\varphi, \boldsymbol{g}_z\} = \left\{\frac{\boldsymbol{a}_r}{h_r}, \frac{\boldsymbol{a}_\varphi}{h_\varphi}, \frac{\boldsymbol{a}_z}{h_z}\right\} = \left\{\boldsymbol{a}_r, \frac{1}{r}\,\boldsymbol{a}_\varphi, \boldsymbol{a}_z\right\} \tag{167}$$

or

$$\boldsymbol{g}_r = (\cos\varphi, \sin\varphi, 0)^\mathsf{T}, \qquad \boldsymbol{g}_\varphi = (-\sin\varphi, \cos\varphi, 0)^\mathsf{T}, \qquad \boldsymbol{g}_z = (0,0,1)^\mathsf{T}. \tag{168}$$

Clearly, these base vectors are orthonormal, i.e., they satisfy $\boldsymbol{g}_i \cdot \boldsymbol{g}_j = \delta_{ij}$.

Using the above definitions of the covariant basis, the *total differential* vector becomes

$$\mathrm{d}\boldsymbol{x} = \frac{\partial \boldsymbol{x}}{\partial \theta^i}\,\mathrm{d}\theta^i = \mathrm{d}\theta^i\,\boldsymbol{a}_i = \mathrm{d}r\,\boldsymbol{a}_r + \mathrm{d}\varphi\,\boldsymbol{a}_\varphi + \mathrm{d}z\,\boldsymbol{a}_z. \tag{169}$$

Furthermore, we can replace the base vectors $\{\boldsymbol{a}_i\}$ by the physical basis $\{\boldsymbol{g}_i\}$ to finally arrive at

$$\mathrm{d}\boldsymbol{x} = \sum_i \frac{\partial \boldsymbol{x}}{\partial \theta^i}\,\mathrm{d}\theta^i = \sum_i \boldsymbol{a}_i\,\mathrm{d}\theta^i = \sum_i \boldsymbol{g}_i\,h_i\,\mathrm{d}\theta^i. \tag{170}$$

Note that in this orthonormal basis we no longer have to differentiate between covariant and contravariant bases, and hence here and in the following we write all indices as subscripts for simplicity. For the example of cylindrical polar coordinates we have

$$\begin{aligned}\mathrm{d}\boldsymbol{x} &= \mathrm{d}r\,\boldsymbol{a}_r + \mathrm{d}\varphi\,\boldsymbol{a}_\varphi + \mathrm{d}z\,\boldsymbol{a}_z = \mathrm{d}r\,h_r\,\boldsymbol{g}_r + \mathrm{d}\varphi\,h_\varphi\,\boldsymbol{g}_\varphi + \mathrm{d}z\,h_z\,\boldsymbol{g}_z \\ &= \mathrm{d}r\,\boldsymbol{g}_r + \mathrm{d}\varphi\,r\,\boldsymbol{g}_\varphi + \mathrm{d}z\,\boldsymbol{g}_z.\end{aligned} \tag{171}$$

Another helpful relation is the differential volume element in an orthonormal basis. In a Cartesian reference frame, the differential volume element is given by $\mathrm{d}v = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$. For (orthogonal) curvilinear coordinates this no longer holds; instead the differential volume element, which we quickly mention without proof, is given by

$$\mathrm{d}v = a\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2\,\mathrm{d}\theta^3, \tag{172}$$

where $a = \sqrt{\det[a_{ij}]}$ is the square root of the determinant of the metric tensor of the (covariant) basis $\{\boldsymbol{a}_i\}$. For cylindrical polar coordinates, the determinant of the metric tensor is $a = \det[a_{ij}] = h_r\,h_\varphi\,h_z = r$. Hence, the differential volume element in this basis is given by

$$\mathrm{d}v = a\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\theta_3 = r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z. \tag{173}$$

Next, let us derive the components of the gradient in our orthogonal coordinate system, which will later allow us to extend our previous definition of the gradient, divergence and curl. By recalling relation (119), we may write the differential of a scalar function $f : \mathbb{R}^d \to \mathbb{R}$ as

$$\mathrm{d}f = \sum_i \frac{\partial f}{\partial \theta^i}\,\mathrm{d}\theta^i = \operatorname{grad} f \cdot \mathrm{d}\boldsymbol{x} = (\operatorname{grad} f)_i\,\boldsymbol{g}_i \cdot \left(\sum_j \boldsymbol{g}_j\,h_j\,\mathrm{d}\theta^j\right), \tag{174}$$

but for an orthonormal basis we know that $\boldsymbol{g}_i \cdot \boldsymbol{g}_j = \delta_{ij}$ and therefore

$$\mathrm{d}f = (\operatorname{grad} f)_i\, \boldsymbol{g}_i \cdot \left(\sum_j \boldsymbol{g}_j\, h_j\, \mathrm{d}\theta^j \right) = \sum_i (\operatorname{grad} f)_i\, h_i\, \mathrm{d}\theta^i. \tag{175}$$

A termwise comparison of the first form in (174) and the final form of (175) gives us the general form of the components of the *gradient* in curvilinear (orthonormal) coordinates:

$$(\operatorname{grad} f)_i = \frac{1}{h_{(i)}} \frac{\partial}{\partial \theta^{(i)}} \tag{176}$$

For convenience, let us also introduce the components of the *del operator* as

$$\nabla_i = (\operatorname{grad} f)_i = \frac{1}{h_{(i)}} \frac{\partial}{\partial \theta^{(i)}}, \tag{177}$$

so that we may write the gradient as

$$\operatorname{grad} f = (\nabla_i f)\, \boldsymbol{g}_i. \tag{178}$$

Let us apply this concept to derive the gradient in cylindrical polar coordinates. We had already seen that for polar coordinates (r, φ, z) we can use relations (163) to arrive at the covariant basis (164). Next, we obtained the components of the metric tensor (165) and read out the scale factors

$$h_r = 1, \qquad h_\varphi = r, \qquad h_z = 1. \tag{179}$$

Therefore, the components of the del operator in the polar basis follow as

$$[\nabla_i] = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial z} \right)^{\mathsf{T}} \tag{180}$$

and the gradient of a scalar field $f : \mathbb{R}^d \to \mathbb{R}$ in cylindrical polar coordinates is

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\boldsymbol{g}_r + \frac{1}{r}\frac{\partial f}{\partial \varphi}\boldsymbol{g}_\varphi + \frac{\partial f}{\partial z}\boldsymbol{g}_z. \tag{181}$$

We are now in place to compute the gradient, divergence and curl in our polar basis by generalizing our previous definitions in the following manner. Let us define for any orthonormal basis

$$\operatorname{grad}(\cdot) = \nabla_i(\cdot) \otimes \boldsymbol{g}_i, \qquad \operatorname{div}(\cdot) = \nabla_i(\cdot) \cdot \boldsymbol{g}_i, \qquad \operatorname{curl}(\cdot) = -\nabla_i(\cdot) \times \boldsymbol{g}_i. \tag{182}$$

Besides replacing $(\cdot)_{,i}$ by $\nabla_i(\cdot)$, this looks identical to (110). However, it requires one additional and very important remark. As we had seen above, the base vectors in curvilinear coordinates (such as in our example of polar coordinates) are no longer constant but we now have $\boldsymbol{g}_i = \boldsymbol{g}_i(\boldsymbol{x})$. Therefore, it is essential to keep in mind that, when writing $\nabla_i(\cdot)$ this implies differentiation of the entire quantity $(\cdot)$ not just its components.

Let us consider the following instructive example: we compute the *divergence of a vector field in cylindrical polar coordinates*:

$$\begin{aligned}\operatorname{div} \boldsymbol{v} = \boldsymbol{\nabla} \cdot \boldsymbol{v} = \nabla_i(\boldsymbol{v}) \cdot \boldsymbol{g}_i = \nabla_i(v_j\, \boldsymbol{g}_j) \cdot \boldsymbol{g}_i &= (\nabla_i v_j)\, \boldsymbol{g}_j \cdot \boldsymbol{g}_i + v_j\, (\nabla_i \boldsymbol{g}_j) \cdot \boldsymbol{g}_i \\ &= (\nabla_i v_i) + v_j\, (\nabla_i \boldsymbol{g}_j) \cdot \boldsymbol{g}_i\end{aligned}$$

The second term in the final equation is new and vanishes in Cartesian coordinates. To compute the divergence, we thus need the derivatives of the base vectors, which can be obtained by differentiation of (168):

$$\begin{array}{lll} \nabla_r \boldsymbol{g}_r = \boldsymbol{g}_{r,r} = 0, & \nabla_\varphi \boldsymbol{g}_r = \frac{1}{r}\boldsymbol{g}_{r,\varphi} = \frac{1}{r}\boldsymbol{g}_\varphi, & \nabla_z \boldsymbol{g}_r = 0, \\ \nabla_r \boldsymbol{g}_\varphi = \boldsymbol{g}_{\varphi,r} = 0, & \nabla_\varphi \boldsymbol{g}_\varphi = \frac{1}{r}\boldsymbol{g}_{\varphi,\varphi} = -\frac{1}{r}\boldsymbol{g}_r, & \nabla_z \boldsymbol{g}_\varphi = \boldsymbol{g}_{\varphi,z} = 0, \\ \nabla_r \boldsymbol{g}_z = \boldsymbol{g}_{z,r} = 0, & \nabla_\varphi \boldsymbol{g}_z = \frac{1}{r}\boldsymbol{g}_{z,\varphi} = 0, & \nabla_z \boldsymbol{g}_z = \boldsymbol{g}_{z,z} = 0. \end{array}$$

Therefore, with $\boldsymbol{g}_r \cdot \boldsymbol{g}_r = 1$ and $\boldsymbol{g}_r \cdot \boldsymbol{g}_\varphi = 0$ we finally arrive at

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= (\nabla_i v_i) + v_j\,(\nabla_i \boldsymbol{g}_j) \cdot \boldsymbol{g}_i = \left(\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z} \right) + v_r \frac{1}{r} \boldsymbol{g}_\varphi \cdot \boldsymbol{g}_\varphi - v_\varphi \frac{1}{r} \boldsymbol{g}_r \cdot \boldsymbol{g}_\varphi \\ &= \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r}. \end{aligned} \tag{183}$$

Similarly, we can compute the *gradient of a vector field in cylindrical polar coordinates*:

$$\begin{aligned} \operatorname{grad} \boldsymbol{v} = \boldsymbol{\nabla} \otimes \boldsymbol{v} &= \nabla_i(\boldsymbol{v}) \otimes \boldsymbol{g}_i = \nabla_i(v_j\,\boldsymbol{g}_j) \otimes \boldsymbol{g}_i = (\nabla_i v_j)\,\boldsymbol{g}_j \otimes \boldsymbol{g}_i + v_j\,(\nabla_i \boldsymbol{g}_j) \otimes \boldsymbol{g}_i \\ &= (\nabla_i v_j)\,\boldsymbol{g}_j \otimes \boldsymbol{g}_i + v_r\,(\nabla_\varphi \boldsymbol{g}_r) \otimes \boldsymbol{g}_\varphi + v_\varphi\,(\nabla_\varphi \boldsymbol{g}_\varphi) \otimes \boldsymbol{g}_\varphi \\ &= (\nabla_i v_j)\,\boldsymbol{g}_j \otimes \boldsymbol{g}_i + \frac{v_r}{r}\,\boldsymbol{g}_\varphi \otimes \boldsymbol{g}_\varphi - \frac{v_\varphi}{r}\,\boldsymbol{g}_r \otimes \boldsymbol{g}_\varphi \end{aligned}$$

which becomes in component form:

$$[\operatorname{grad} \boldsymbol{v}] = \begin{bmatrix} v_{r,r} & \frac{1}{r}(v_{r,\varphi} - v_\varphi) & v_{r,z} \\ v_{\varphi,r} & \frac{1}{r}(v_{\varphi,\varphi} + v_r) & v_{\varphi,z} \\ v_{z,r} & \frac{1}{r}v_{z,\varphi} & v_{z,z} \end{bmatrix} \tag{184}$$

In an analogous fashion, the gradient, divergence and curl of any tensor quantity of arbitrary order can be obtained by following the above relations.

We can make use of the fact that, for computing the above differential forms, one always needs the same derivatives of the base vectors. To this end, we alternative express the covariant derivative as follows (but we will not go into much detail). Let us introduce the *Christoffel symbol of the second kind*, which is defined by

$$\Gamma^k_{ij} = \frac{1}{2}\,g^{km}\,(g_{mi,j} + g_{mj,i} - g_{ij,m})\,. \tag{185}$$

Here, $[g^{ij}]$ is the inverse of the metric tensor $[g_{ij}]$. Furthermore, we define the *physical Christoffel symbol*

$$\tilde{\Gamma}^k_{ij} = \frac{h_{(k)}}{h_{(i)}\,h_{(j)}}\,\Gamma^{(k)}_{(i)(j)} - \frac{h_{(i),(j)}}{h_{(i)}\,h_{(j)}}\,\delta_{(i)(k)}. \tag{186}$$

(The parentheses enforce that no index be summed.) Those definitions admit the concise relation

$$\nabla_k\,\boldsymbol{g}_j = -\tilde{\Gamma}^j_{ik}\,\boldsymbol{g}_i. \tag{187}$$

For example, the gradient of a (differentiable) vector field $\boldsymbol{u} : \mathbb{R}^d \to \mathbb{R}^d$ can now be expressed as

$$\begin{aligned} \operatorname{grad} \boldsymbol{u} &= \nabla_j(u_i\,\boldsymbol{g}_i) \otimes \boldsymbol{g}_j = \nabla_j u_i\,\boldsymbol{g}_i \otimes \boldsymbol{g}_j + u_i\,(\nabla_j \boldsymbol{g}_i) \otimes \boldsymbol{g}_j \\ &= \nabla_j u_i\,\boldsymbol{g}_i \otimes \boldsymbol{g}_j - u_i\,\tilde{\Gamma}^i_{kj}\,\boldsymbol{g}_k \otimes \boldsymbol{g}_j = \left(\nabla_j u_i - u_l\,\tilde{\Gamma}^l_{ij}\right)\boldsymbol{g}_i \otimes \boldsymbol{g}_j \end{aligned} \tag{188}$$

and the divergence of a (differentiable) vector field $\boldsymbol{u} : \mathbb{R}^d \to \mathbb{R}^d$ as

$$\begin{aligned}\operatorname{div} \boldsymbol{u} &= \nabla_j (u_i \, \boldsymbol{g}_i) \cdot \boldsymbol{g}_j = \nabla_j u_i \, \boldsymbol{g}_i \cdot \boldsymbol{g}_j + u_i \, (\nabla_j \boldsymbol{g}_i) \cdot \boldsymbol{g}_j \\ &= \nabla_j u_i \, \delta_{ij} - u_i \, \tilde{\Gamma}^i_{kj} \, \boldsymbol{g}_k \cdot \boldsymbol{g}_j = \nabla_i u_i - u_l \, \tilde{\Gamma}^l_{jj}.\end{aligned} \tag{189}$$

As before, the first term in each of these final expressions arises from the spatially-varying components $v_i(\boldsymbol{x})$ of the vector field, just like in Cartesian coordinates as discussed before. The second term stems from the spatially-varying base vectors $\boldsymbol{g}_i(\boldsymbol{x})$ and it vanishes in a fixed Cartesian coordinate system.

Let us demonstrate this concept by means of an example: we return to cylindrical polar coordinates and compute the divergence of a (differentiable) vector field $\boldsymbol{v} : \mathbb{R}^d \to \mathbb{R}^d$. For cylindrical polar coordinates, the metric tensor and its inverse are

$$[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad [g^{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{190}$$

By application of (185), we obtain the Christoffel symbols (these are simple to compute in this case because most components vanish due to the specific form of the metric tensors):

$$[\Gamma^r_{ij}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad [\Gamma^\varphi_{ij}] = \begin{pmatrix} 0 & 1/r & 0 \\ 1/r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad [\Gamma^z_{ij}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{191}$$

and the physical Christoffel symbols follow from (186):

$$[\tilde{\Gamma}^r_{ij}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1/r & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad [\tilde{\Gamma}^\varphi_{ij}] = \begin{pmatrix} 0 & 1/r & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad [\tilde{\Gamma}^z_{ij}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{192}$$

Therefore, the divergence of a vector field $\boldsymbol{v}$ in cylindrical polar coordinates given by (189) is obtained as follows:

$$\begin{aligned}\operatorname{div} \boldsymbol{v} &= \nabla_i \, v_i - v_l \, \tilde{\Gamma}^l_{jj} = v_{r,r} + \frac{1}{r} v_{\varphi,\varphi} + v_{z,z} - v_r \, \tilde{\Gamma}^r_{\varphi\varphi} \\ &= \frac{\partial \, v_r}{\partial \, r} + \frac{1}{r} \frac{\partial \, v_\varphi}{\partial \, \varphi} + \frac{\partial \, v_z}{\partial \, z} + \frac{v_r}{r},\end{aligned} \tag{193}$$

which agrees with our previous result (183). The major benefit of the Christoffel symbols lies in the fact that their use only requires the computation of the metric tensor and its derivatives instead of differentiating each unit vector individually.

2.3 Tensor Transformation Rules

Now that we have introduced various coordinate systems, it will be helpful to study the relations between the components of vectors and tensors when switching between different coordinate systems. For simplicity, in the sequel we only consider orthonormal bases (where there is no difference between covariant and contravariant formulation); this includes Cartesian and polar coordinates. Let us start by introducing two different bases $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3\}$ and $\{\tilde{\boldsymbol{g}}_1, \tilde{\boldsymbol{g}}_2, \tilde{\boldsymbol{g}}_3\}$.

Suppose, e.g., that we want to rotate our coordinate system from the original basis $\{\boldsymbol{g}_i\}$ to the new basis $\{\tilde{\boldsymbol{g}}_i\}$. We can express the position vector of any point $\boldsymbol{x}$ with respect to both bases:

$$\boldsymbol{x} = x_i\,\boldsymbol{g}_i = \tilde{x}_i\,\tilde{\boldsymbol{g}}_i \tag{194}$$

Recall that we introduced an alternative definition of the bases vectors: $\boldsymbol{g}_i = \partial\boldsymbol{x}/\partial x_i$ and analogously $\tilde{\boldsymbol{g}}_i = \partial\boldsymbol{x}/\partial\tilde{x}_i$. This can be used to obtain the transformation relation by application of the chain rule:

$$\boldsymbol{g}_i = \frac{\partial\boldsymbol{x}}{\partial x_i} = \frac{\partial\boldsymbol{x}}{\partial\tilde{x}_j}\frac{\partial\tilde{x}_j}{\partial x_i} = \tilde{\boldsymbol{g}}_j\frac{\partial\tilde{x}_j}{\partial x_i} \tag{195}$$

Alternatively, we can apply the same concept to the second basis:

$$\tilde{\boldsymbol{g}}_i = \frac{\partial\boldsymbol{x}}{\partial\tilde{x}_i} = \frac{\partial\boldsymbol{x}}{\partial x_j}\frac{\partial x_j}{\partial\tilde{x}_i} = \boldsymbol{g}_j\frac{\partial x_j}{\partial\tilde{x}_i} \tag{196}$$

This gives the transformation rules for base vectors (so-called *covariant transformation*):

$$\boldsymbol{g}_i = \tilde{\boldsymbol{g}}_j\frac{\partial\,\tilde{x}_j}{\partial\,x_i}, \qquad\qquad \tilde{\boldsymbol{g}}_i = \boldsymbol{g}_j\frac{\partial\,x_j}{\partial\,\tilde{x}_i} \tag{197}$$

We can use the same concept to derive transformation rules for the components of vectors (as the base vectors change, the coordinates of points with respect to the changing basis must change as well). To this end, we write, using the above transformation rule,

$$\boldsymbol{v} = v_i\,\boldsymbol{g}_i = \tilde{v}_i\,\tilde{\boldsymbol{g}}_i = \tilde{v}_i\frac{\partial\,x_j}{\partial\,\tilde{x}_i}\,\boldsymbol{g}_j \qquad\Rightarrow\qquad v_i = \tilde{v}_j\frac{\partial x_i}{\partial\tilde{x}_j}. \tag{198}$$

This gives the *contravariant transformation rules* for vector components:

$$v_i = \tilde{v}_j\frac{\partial x_i}{\partial\tilde{x}_j}, \qquad\qquad \tilde{v}_i = v_j\frac{\partial\tilde{x}_i}{\partial x_j} \tag{199}$$

Writing out the partial derivatives is cumbersome. Instead, it is convenient to introduce a *transformation matrix* $\boldsymbol{Q}$ which describes the mapping between the two bases and whose components are defined by

$$Q_{ij} = \frac{\partial\,x_i}{\partial\,\tilde{x}_j} \qquad\Rightarrow\qquad v_i = Q_{ij}\,\tilde{v}_j. \tag{200}$$

An even simpler interpretation for the components of $\boldsymbol{Q}$ follows from the following observation:

$$\boldsymbol{v} = v_i\,\boldsymbol{g}_i = \tilde{v}_i\,\tilde{\boldsymbol{g}}_i \qquad\Rightarrow\qquad v_j = (v_i\,\boldsymbol{g}_i)\cdot\boldsymbol{g}_j = (\tilde{\boldsymbol{g}}_i\cdot\boldsymbol{g}_j)\tilde{v}_i \tag{201}$$

and therefore (renaming indices)

$$Q_{ij} = \boldsymbol{g}_i\cdot\tilde{\boldsymbol{g}}_j. \tag{202}$$

Similarly, we can perform the inverse transformation. We know that

$$\delta_{ik} = \frac{\partial\,\tilde{x}_i}{\partial\,\tilde{x}_k} = \frac{\partial\,\tilde{x}_i}{\partial\,x_j}\frac{\partial\,x_j}{\partial\,\tilde{x}_k} = \frac{\partial\,\tilde{x}_i}{\partial\,x_j}Q_{jk} \tag{203}$$

But we also know that $\delta_{ik} = Q^{-1}_{ij} Q_{jk}$, so that we recover the inverse transformation matrix

$$Q^{-1}_{ij} = \frac{\partial \tilde{x}_i}{\partial x_j}. \tag{204}$$

The transformation matrix admits a cleaner and shorter formulation of the transformation rules introduced above, and its components can easily be computed from (202). We note that $\boldsymbol{Q}$ is not an ordinary second-order tensor because it does not come with a basis, which is why we refer to it as the transformation matrix.

As a further generalization, we can apply the obtained transformation rules not only to vectors but to abritrary higher-order tensors. Consider e.g. a second-order tensor $\boldsymbol{T}$ which can be expressed as

$$\boldsymbol{T} = T_{ij}\, \boldsymbol{g}_i \otimes \boldsymbol{g}_j = \tilde{T}_{ij}\, \tilde{\boldsymbol{g}}_i \otimes \tilde{\boldsymbol{g}}_j. \tag{205}$$

In order to transform to the new basis, we simply apply the above transformation rules for vectors independently to each of the two base vectors. In this way, we can transform any tensor of arbitrary order by the general transformation rule

$$T_{ij\dots n} = Q_{ia}\, Q_{ib} \dots Q_{nd}\, \tilde{T}_{ab\dots d}. \tag{206}$$

The complete set of transformation rules for vectors and second-order tensors (in indicial and symbolic notation) now reads:

$$\begin{array}{ll} u_i = Q_{ij}\, \tilde{u}_j & \boldsymbol{u} = \boldsymbol{Q}\tilde{\boldsymbol{u}} \\ \tilde{u}_i = Q^{-1}_{ij}\, u_j & \tilde{\boldsymbol{u}} = \boldsymbol{Q}^{-1}\boldsymbol{u} \\ T_{ij} = Q_{ik}\, \tilde{T}_{kl}\, Q_{jl} & \boldsymbol{T} = \boldsymbol{Q}\tilde{\boldsymbol{T}}\boldsymbol{Q}^{\mathsf{T}} \\ \tilde{T}_{ij} = Q^{-1}_{ik}\, T_{kl}\, Q^{-1}_{jl} & \tilde{\boldsymbol{T}} = \boldsymbol{Q}^{-1}\boldsymbol{T}\boldsymbol{Q}^{-T} \end{array} \tag{207}$$

These transformation rules become trivial when applied to basis *rotations*, as we show in the following example. Assume that our original coordinate system is given by a two-dimensional Cartesian basis $\{\boldsymbol{g}_1, \boldsymbol{g}_2\}$, and we want to rotate our coordinate system by an angle φ (counter-clockwise) so that

$$\tilde{\boldsymbol{g}}_1 = \boldsymbol{g}_1 \cos\varphi + \boldsymbol{g}_2 \sin\varphi, \qquad \tilde{\boldsymbol{g}}_2 = -\boldsymbol{g}_1 \sin\varphi + \boldsymbol{g}_2 \cos\varphi. \tag{208}$$

The components of the transformation matrix are obtained from

$$[Q_{ij}] = [\boldsymbol{g}_i \cdot \tilde{\boldsymbol{g}}_j] = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}. \tag{209}$$

We see that in this case the transformation matrix is constant and, for the special case of a rotation, $\boldsymbol{Q}$ is a rotation tensor with $\boldsymbol{Q} \in SO(2)$. The inverse is computed as

$$[Q^{-1}_{ij}] = [\boldsymbol{g}_i \cdot \tilde{\boldsymbol{g}}_j] = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} = [Q^{\mathsf{T}}_{ij}]. \tag{210}$$

This could have been expected since $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^{\mathsf{T}}$ because $\boldsymbol{Q} \in SO(2)$.

For example, consider a point in space described by coordinates $(v_i) = (1,1)^{\mathsf{T}}$ in the original basis. We rotate our coordinate system by a counter-clockwise angle φ and compute the transformed coordinates:

$$\tilde{v}_i = Q^{-1}_{ij} v_j = (\cos\varphi + \sin\varphi,\ \cos\varphi - \sin\varphi)^{\mathsf{T}}. \tag{211}$$

For example, for a rotation by $\varphi = \pi/4$ we obtain

$$\tilde{v}_i = \left(\sqrt{2},\ 0\right)^\mathsf{T} \tag{212}$$

which is the correct result.

In continuum mechanics, we often apply the transformation rules to second-order tensors such as the infinitesimal strain and stress tensors whose components must be computed in a rotated coordinate system. Such a coordinate roation can now performed easily by the above relations. In particular, the stress tensor transforms by (using the fact that $\boldsymbol{Q}$ is orthogonal)

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{Q}^{-1}\boldsymbol{\sigma}\boldsymbol{Q}^{-\mathsf{T}} = \boldsymbol{Q}^T\boldsymbol{\sigma}\boldsymbol{Q}. \tag{213}$$

Also, we can verify that rotating the obtained tensor $\tilde{\boldsymbol{\sigma}}$ back by the same angle into the opposite direction (i.e. applying the inverse transformation matrix $\boldsymbol{Q}^{-1}$) yields

$$(\boldsymbol{Q}^{-1})^{-1}\tilde{\boldsymbol{\sigma}}(\boldsymbol{Q}^{-1})^{-\mathsf{T}} = \boldsymbol{Q}\tilde{\boldsymbol{\sigma}}\boldsymbol{Q}^\mathsf{T} = \boldsymbol{Q}(\boldsymbol{Q}^T\boldsymbol{\sigma}\boldsymbol{Q})\boldsymbol{Q}^\mathsf{T} = \boldsymbol{\sigma}, \tag{214}$$

i.e. we correctly arrived at the original unrotated tensor.

Summary of Indicial Notation

Useful expressions in indicial notation in Cartesian coordinates:

symbolic notation	full component form	indicial notation
$\boldsymbol{v}$	$v_i \boldsymbol{g}_i$	$[\boldsymbol{v}]_i = v_i$
$\boldsymbol{v} \cdot \boldsymbol{w}$	$v_i w_i$	$v_i w_i$
$\boldsymbol{v} \otimes \boldsymbol{w}$	$v_i w_j \boldsymbol{g}_i \otimes \boldsymbol{g}_j$	$[\boldsymbol{v} \otimes \boldsymbol{w}]_{ij} = v_i w_j$
$\boldsymbol{v} \times \boldsymbol{w}$	$v_i w_j \epsilon_{ijk} \boldsymbol{g}_k$	$[\boldsymbol{v} \times \boldsymbol{w}]_k = v_i w_j \epsilon_{ijk}$
$\boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{x})$	$v_i w_j x_k \epsilon_{ijk}$	$v_i w_j x_k \epsilon_{ijk}$
$\boldsymbol{T}$	$T_{ij} \boldsymbol{g}_i \otimes \boldsymbol{g}_j$	$[\boldsymbol{T}]_{ij} = T_{ij}$
$\boldsymbol{T}^\mathsf{T}$	$T_{ji} \boldsymbol{g}_i \otimes \boldsymbol{g}_j$	$[\boldsymbol{T}^\mathsf{T}]_{ij} = T_{ji}$
$\boldsymbol{T}\boldsymbol{v}$	$T_{ij} v_j \boldsymbol{g}_i$	$[\boldsymbol{T}\boldsymbol{v}]_i = T_{ij} v_j$
$\boldsymbol{T}^\mathsf{T}\boldsymbol{v}$	$T_{ji} v_j \boldsymbol{g}_i$	$[\boldsymbol{T}^\mathsf{T}\boldsymbol{v}]_i = T_{ji} v_j$
$\boldsymbol{v} \cdot \boldsymbol{T}\boldsymbol{w}$	$v_i T_{ij} w_j$	$v_i T_{ij} w_j$
$\boldsymbol{T}\boldsymbol{S}$	$T_{ij} S_{jk} \boldsymbol{g}_i \otimes \boldsymbol{g}_k$	$[\boldsymbol{T}\boldsymbol{S}]_{ik} = T_{ij} S_{jk}$
$\boldsymbol{T}^\mathsf{T}\boldsymbol{S}$	$T_{ji} S_{jk} \boldsymbol{g}_i \otimes \boldsymbol{g}_k$	$[\boldsymbol{T}^\mathsf{T}\boldsymbol{S}]_{ik} = T_{ji} S_{jk}$
$\boldsymbol{T}\boldsymbol{S}^\mathsf{T}$	$T_{ij} S_{kj} \boldsymbol{g}_i \otimes \boldsymbol{g}_k$	$[\boldsymbol{T}\boldsymbol{S}^\mathsf{T}]_{ik} = T_{ij} S_{kj}$
$\boldsymbol{T} \cdot \boldsymbol{S}$	$T_{ij} S_{ij}$	$T_{ij} S_{ij}$
$\boldsymbol{T} \times \boldsymbol{v}$	$T_{ij} v_k \epsilon_{jkl} \boldsymbol{g}_i \otimes \boldsymbol{g}_l$	$[\boldsymbol{T} \times \boldsymbol{v}]_{il} = T_{ij} v_k \epsilon_{jkl}$
$\boldsymbol{T} \otimes \boldsymbol{v}$	$T_{ij} v_k \boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}_k$	$[\boldsymbol{T} \otimes \boldsymbol{v}]_{ijk} = T_{ij} v_k$
$\boldsymbol{v} \times \boldsymbol{T}$	$v_j T_{ki} \epsilon_{jkl} \boldsymbol{g}_i \otimes \boldsymbol{g}_l$	$[\boldsymbol{v} \times \boldsymbol{T}]_{il} = v_j T_{ki} \epsilon_{jkl}$
$\boldsymbol{v} \otimes \boldsymbol{T}$	$v_i T_{jk} \boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}_k$	$[\boldsymbol{v} \otimes \boldsymbol{T}]_{ijk} = v_i T_{jk}$
$\operatorname{tr} \boldsymbol{T}$	T_{ii}	T_{ii}
$\det(\boldsymbol{T})$	$\epsilon_{ijk} T_{i1} T_{j2} T_{k3}$	$\epsilon_{ijk} T_{i1} T_{j2} T_{k3}$
$\operatorname{grad} \phi$	$\phi_{,i} \boldsymbol{g}_i$	$[\operatorname{grad} \phi]_i = \phi_{,i}$
$\operatorname{grad} \boldsymbol{v}$	$v_{i,j} \boldsymbol{g}_i \otimes \boldsymbol{g}_j$	$[\operatorname{grad} \boldsymbol{v}]_{ij} = v_{i,j}$
$\operatorname{grad} \boldsymbol{T}$	$T_{ij,k} \boldsymbol{g}_i \otimes \boldsymbol{g}_j \otimes \boldsymbol{g}_k$	$[\operatorname{grad} \boldsymbol{T}]_{ijk} = T_{ij,k}$
$\operatorname{div} \boldsymbol{v}$	$v_{i,i}$	$v_{i,i}$
$\operatorname{div} \boldsymbol{T}$	$T_{ij,j} \boldsymbol{g}_i$	$[\operatorname{div} \boldsymbol{T}]_i = T_{ij,j}$
$\operatorname{curl} \boldsymbol{v}$	$v_{i,j} \epsilon_{jik} \boldsymbol{g}_k$	$[\operatorname{curl} \boldsymbol{v}]_k = v_{i,j} \epsilon_{jik}$
$\operatorname{curl} \boldsymbol{T}$	$T_{ik,l} \epsilon_{lkj} \boldsymbol{g}_i \otimes \boldsymbol{g}_j$	$[\operatorname{curl} \boldsymbol{T}]_{ij} = T_{ik,l} \epsilon_{lkj}$
$\operatorname{div}(\operatorname{grad} \phi) = \boldsymbol{\Delta}\phi$	$\phi_{,kk} \boldsymbol{g}_i$	$[\operatorname{div}(\operatorname{grad} \phi)]_i = \phi_{,kk}$
$\operatorname{div}(\operatorname{grad} \boldsymbol{v}) = \boldsymbol{\Delta}\boldsymbol{v}$	$v_{i,kk} \boldsymbol{g}_i$	$[\operatorname{div}(\operatorname{grad} \boldsymbol{v})]_i = v_{i,kk}$

Identities involving the dyads:

$$(\boldsymbol{v} \otimes \boldsymbol{w})\boldsymbol{u} = \boldsymbol{v}\,(\boldsymbol{w} \cdot \boldsymbol{u}) \qquad \alpha\, \boldsymbol{a} \otimes \boldsymbol{b} = (\alpha\, \boldsymbol{a}) \otimes \boldsymbol{b} = \boldsymbol{a} \otimes (\alpha \boldsymbol{b})$$

$$(\boldsymbol{u} \otimes \boldsymbol{v}) \times \boldsymbol{w} = \boldsymbol{u}\,(\boldsymbol{v} \times \boldsymbol{w}) \qquad \alpha\,(\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{w} = \alpha\,(\boldsymbol{w} \otimes \boldsymbol{v})\boldsymbol{u}$$

$$(\boldsymbol{v} + \boldsymbol{w}) \otimes \boldsymbol{u} = \boldsymbol{v} \otimes \boldsymbol{u} + \boldsymbol{w} \otimes \boldsymbol{u} \qquad (\boldsymbol{u} \otimes \boldsymbol{v})\boldsymbol{T} = \boldsymbol{u} \otimes \boldsymbol{T}^\mathsf{T}\boldsymbol{v}$$

$$\boldsymbol{u} \otimes (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \otimes \boldsymbol{v} + \boldsymbol{u} \otimes \boldsymbol{w} \qquad \boldsymbol{T}(\boldsymbol{u} \otimes \boldsymbol{v}) = (\boldsymbol{T}\boldsymbol{u}) \otimes \boldsymbol{v}$$

Definition of gradient, divergence and curl in an orthonormal basis:

$$\operatorname{grad}(\cdot) = \nabla_i(\cdot) \otimes \boldsymbol{g}_i, \qquad \operatorname{div}(\cdot) = \nabla_i(\cdot) \cdot \boldsymbol{g}_i, \qquad \operatorname{curl}(\cdot) = -\nabla_i(\cdot) \times \boldsymbol{g}_i$$